What is a random medium ?

Problem: Wave propagation in a highly heterogeneous medium.

Stochastic modeling: the medium is a realization of a random medium (a set of possible media described statistically).
- takes into account the available data (mean, standard deviation of the fluctuations, ...)
- completes the modeling by a statistical description (Gaussian process, ...).

Statistical distribution of the random medium $\Longrightarrow$ statistical distribution of the wave (highly nonlinear problem).

Importance of scaled regimes and asymptotic theory.

Statistical stability (self-averaging property): certain quantities depend only on the statistics of the random medium, not on the particular realization.

Methodology

- Modeling:
  - Identification of the phenomena and equations.
  - Statistical description of the medium parameters.
  - Determination of the scales.
- Asymptotics:
  - Separation of scales.
  - Limit theorems.
- Limit problem:
  - Analysis of the physically relevant quantities.
  - Use of stochastic calculus.

Limit theorems

Law of Large Numbers.

Let $(X_n)_{n\geq 0}$ be independent and identically distributed (i.i.d.) random variables. If $\mathbb{E}[|X_1|] < \infty$, then

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + ... + X_n) \stackrel{n\to\infty}{\longrightarrow} m \text{ almost surely, with } m = \mathbb{E}[X_1]$$

"The empirical mean converges to the statistical mean".

Central Limit Theorem. Fluctuations theory.

Let $(X_n)_{n\geq 0}$ be i.i.d. random variables. If $\mathbb{E}[X_1^2] < \infty$, then

$$\sqrt{n}\left(\bar{X}_n - m\right) = \sqrt{n}\left(\frac{1}{n}(X_1 + X_2 + ... + X_n) - m\right) \stackrel{n\to\infty}{\longrightarrow} \mathcal{N}(0,\sigma^2) \text{ in law}$$

where $\begin{cases} m = \mathbb{E}[X_1] \\ \sigma^2 = \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 = \mathbb{E}[(X_1 - \mathbb{E}[X_1])^2] \end{cases}$

"For large n, the error $\bar{X}_n - m$ obeys a Gaussian distribution $\mathcal{N}(0,\sigma^2/n)$."

The acoustic wave equations

The acoustic pressure $p(z,t)$ and velocity $u(z,t)$ satisfy the continuity and momentum equations

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$
$$\frac{\partial p}{\partial t} + \kappa \frac{\partial u}{\partial z} = 0$$

where $\rho(z)$ is the material density,

$\kappa(z)$ is the bulk modulus of the medium.

Propagation in homogeneous medium

Linear hyperbolic system with ρ, κ constant.

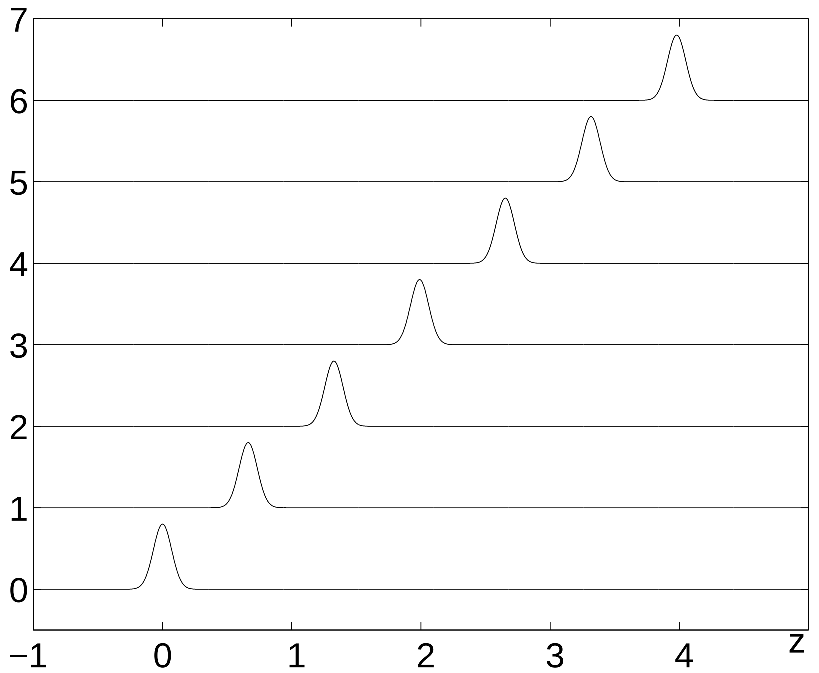
Impedance: $\zeta = \sqrt{\rho\kappa}$. Sound speed: $c = \sqrt{\kappa/\rho}$.

Right- and left-going modes:

$$A = \zeta^{1/2}u + \zeta^{-1/2}p, \qquad B = \zeta^{1/2}u - \zeta^{-1/2}p$$

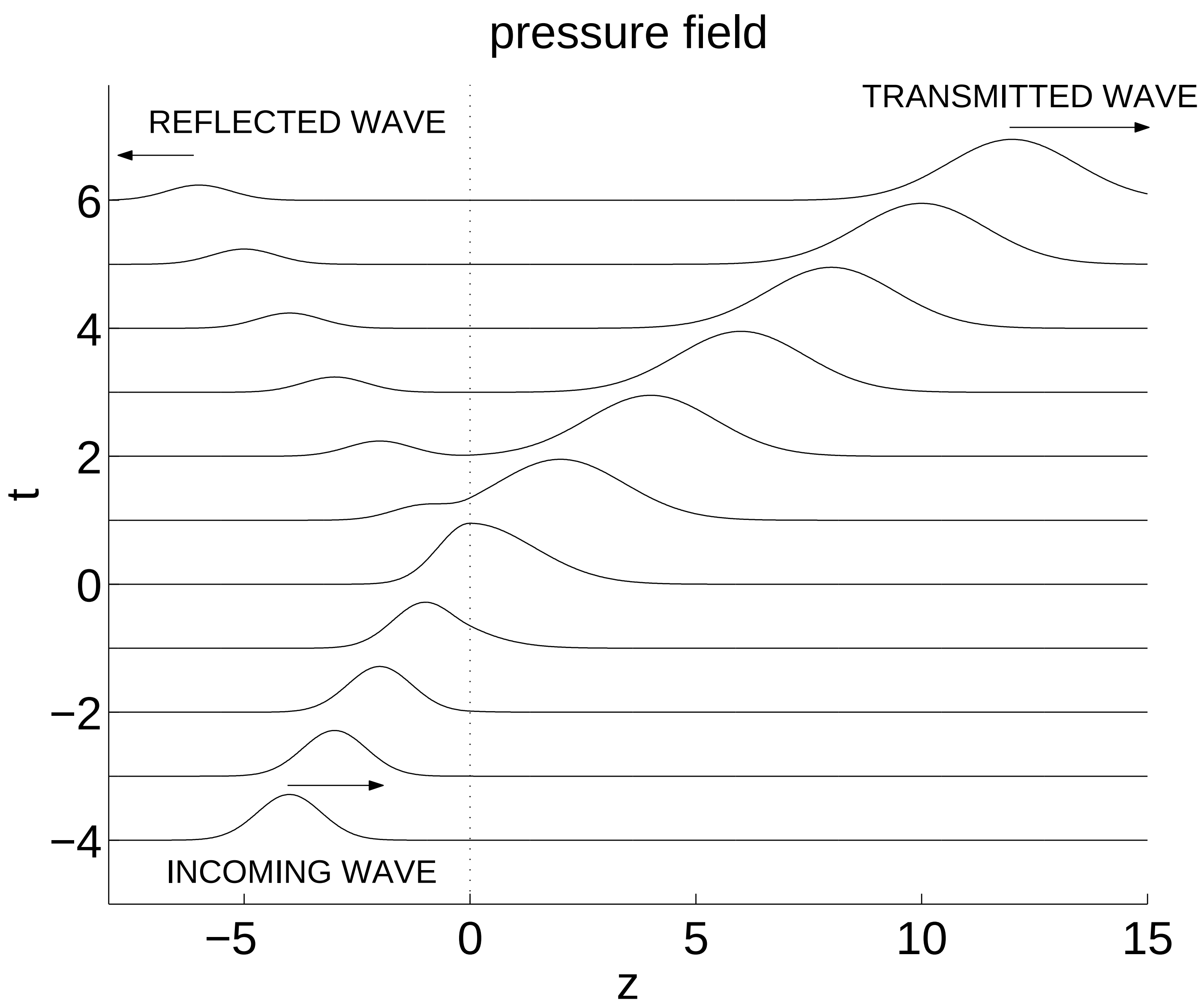
$$\frac{\partial A}{\partial t} + c\frac{\partial A}{\partial z} = 0, \qquad \frac{\partial B}{\partial t} - c\frac{\partial B}{\partial z} = 0$$

A: right-going wave $\qquad$ B: left-going wave.



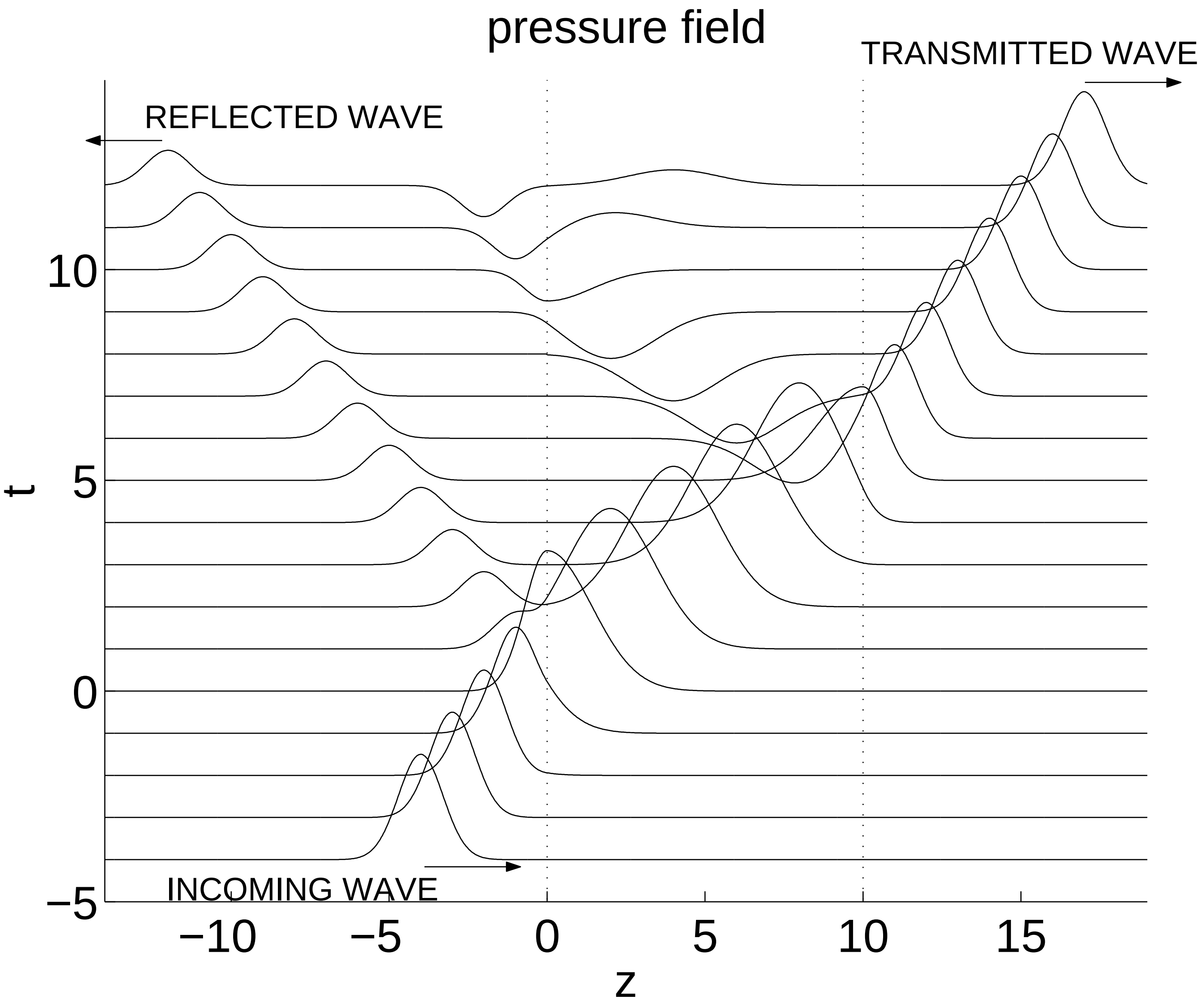
Spatial profiles of the wave at different times for a pure right-going wave

Propagation through an interface



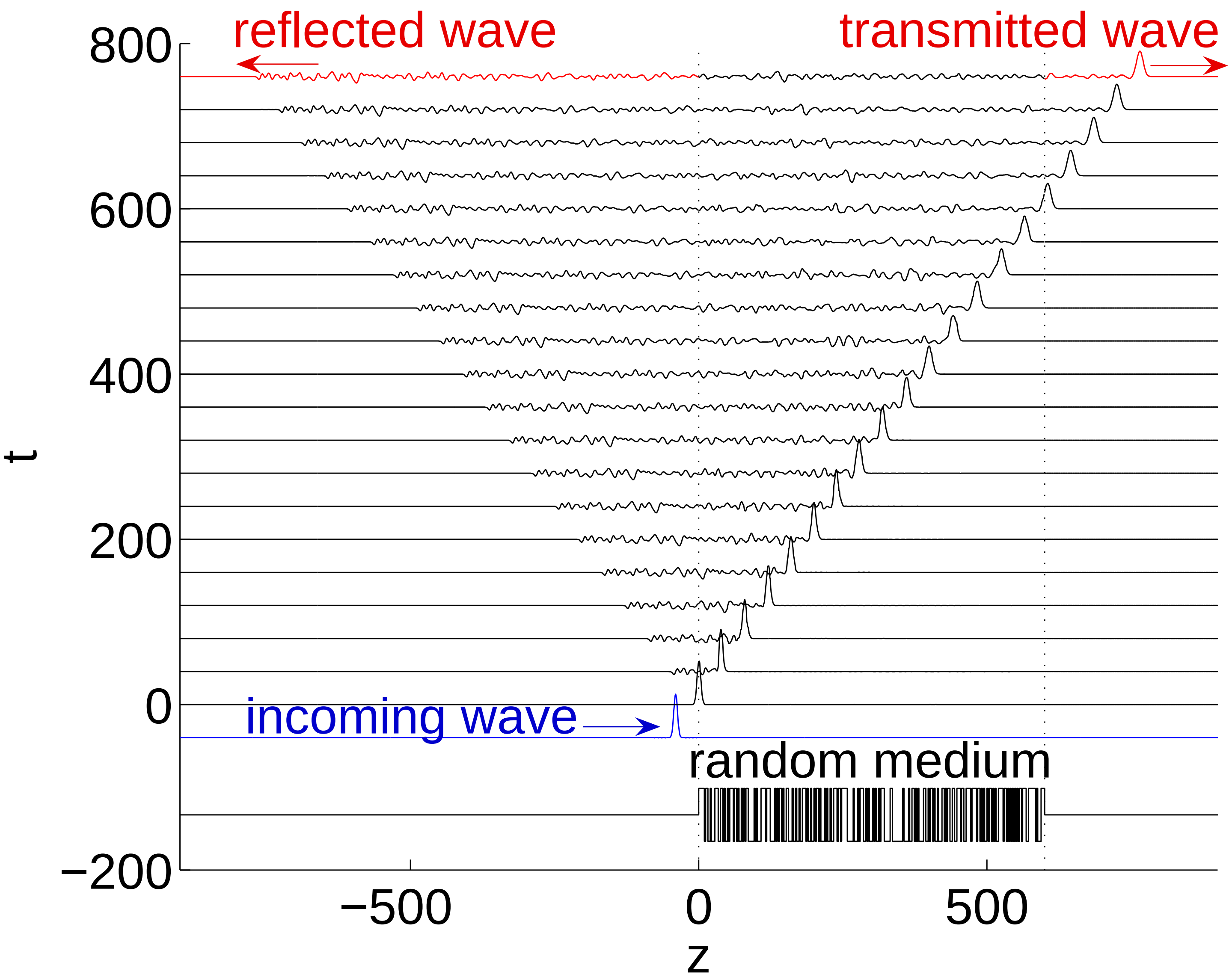
Medium $z < 0$: $c = 1$, $\zeta = 1$. Medium $z > 0$: $c = 2$, $\zeta = 2$.

Propagation through a thick layer



Medium $\left\{ \begin{matrix} z < 0 \\ z > 10 \end{matrix} \right\}$: $c = 1$, $\zeta = 1$. Medium $0 < z < 10$: $c = 2$, $\zeta = 2$.

A numerical experiment in random medium



Random medium: stack of thin layers composed of two materials.

The three scales

The acoustic pressure $p(z,t)$ and velocity $u(z,t)$ satisfy the continuity and momentum equations

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$
$$\frac{\partial p}{\partial t} + \kappa \frac{\partial u}{\partial z} = 0$$

where $\rho(z)$ is the material density,

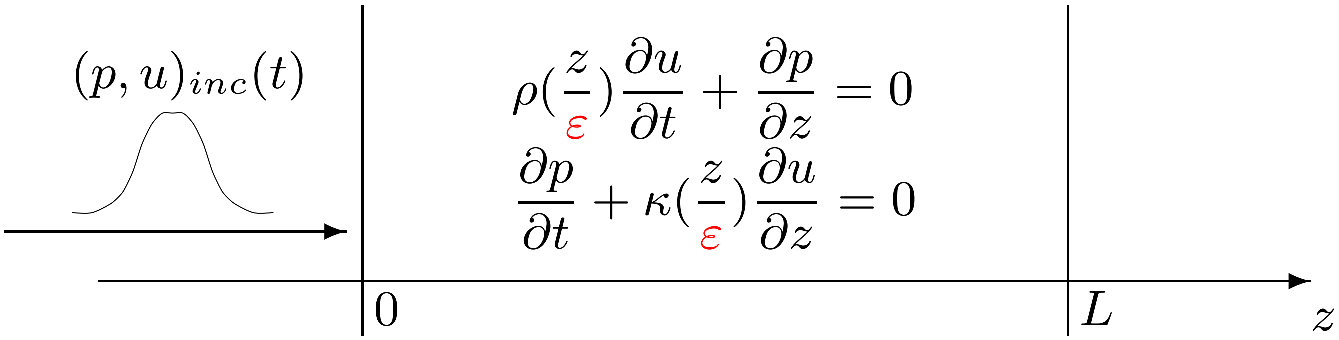
$\kappa(z)$ is the bulk modulus of the medium.

Three scales:

l_c: correlation radius of the random process ρ and κ.

λ: typical wavelength of the incoming pulse.

L: propagation distance.



$$\rho(\frac{z}{\varepsilon})\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$
$$\frac{\partial p}{\partial t} + \kappa(\frac{z}{\varepsilon})\frac{\partial u}{\partial z} = 0$$

Model: $\rho = \rho(z/\varepsilon)$ and $\kappa = \kappa(z/\varepsilon)$, where $0 < \varepsilon \ll 1$ and ρ, κ are random functions.

Perform a Fourier transform with respect to t:

$$u(t, z) = \frac{1}{2\pi}\int \hat{u}(\omega, z)e^{-i\omega t}d\omega, \quad p(t, z) = \frac{1}{2\pi}\int \hat{p}(\omega, z)e^{-i\omega t}d\omega$$

so that we get a system of ordinary differential equations:

$$\frac{d\boldsymbol{X}^\varepsilon}{dz} = \boldsymbol{F}(\frac{z}{\varepsilon}, \boldsymbol{X}^\varepsilon),$$

where

$$\boldsymbol{X}^\varepsilon = \begin{pmatrix} \hat{p} \\ \hat{u} \end{pmatrix}, \quad \boldsymbol{F}(z, \boldsymbol{X}) = -i\omega \begin{pmatrix} 0 & \rho(z) \\ \frac{1}{\kappa(z)} & 0 \end{pmatrix} \boldsymbol{X}$$

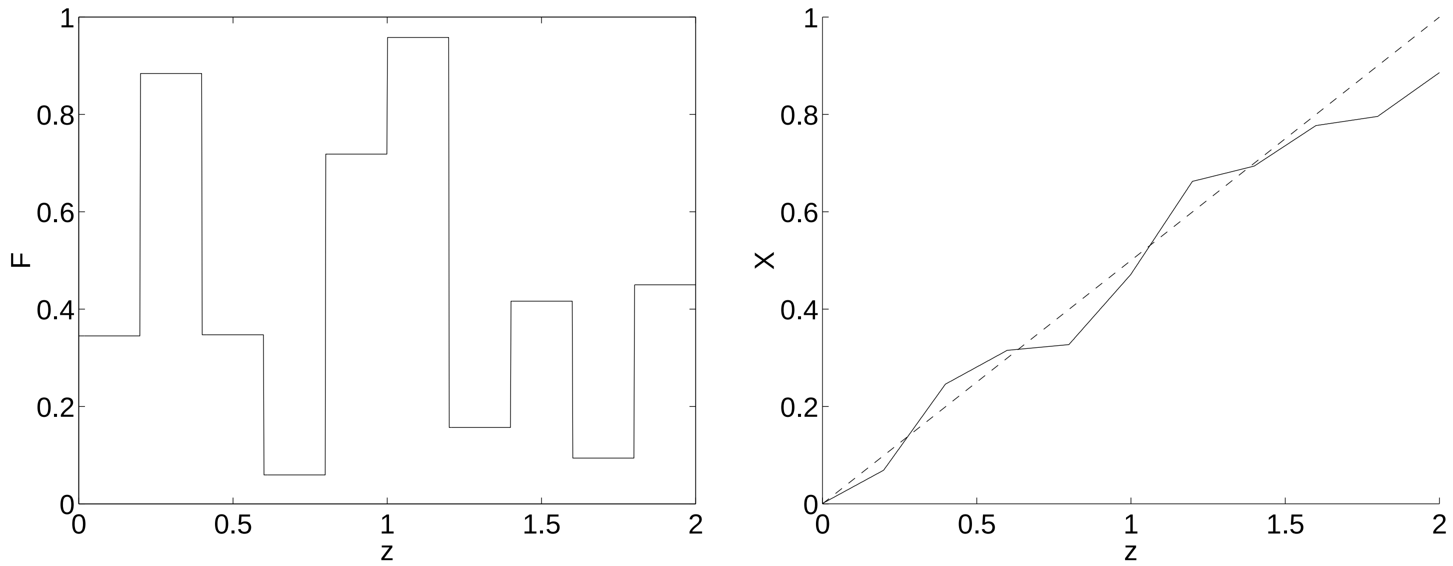
Method of averaging: Toy model

Let $X^\varepsilon(z) \in \mathbb{R}$ be the solution of

$$\frac{dX^\varepsilon}{dz} = F\left(\frac{z}{\varepsilon}\right)$$

with $F(z) = \sum_{i=1}^\infty F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = \bar{F}$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$.

($z \mapsto t$, particle in a random velocity field)

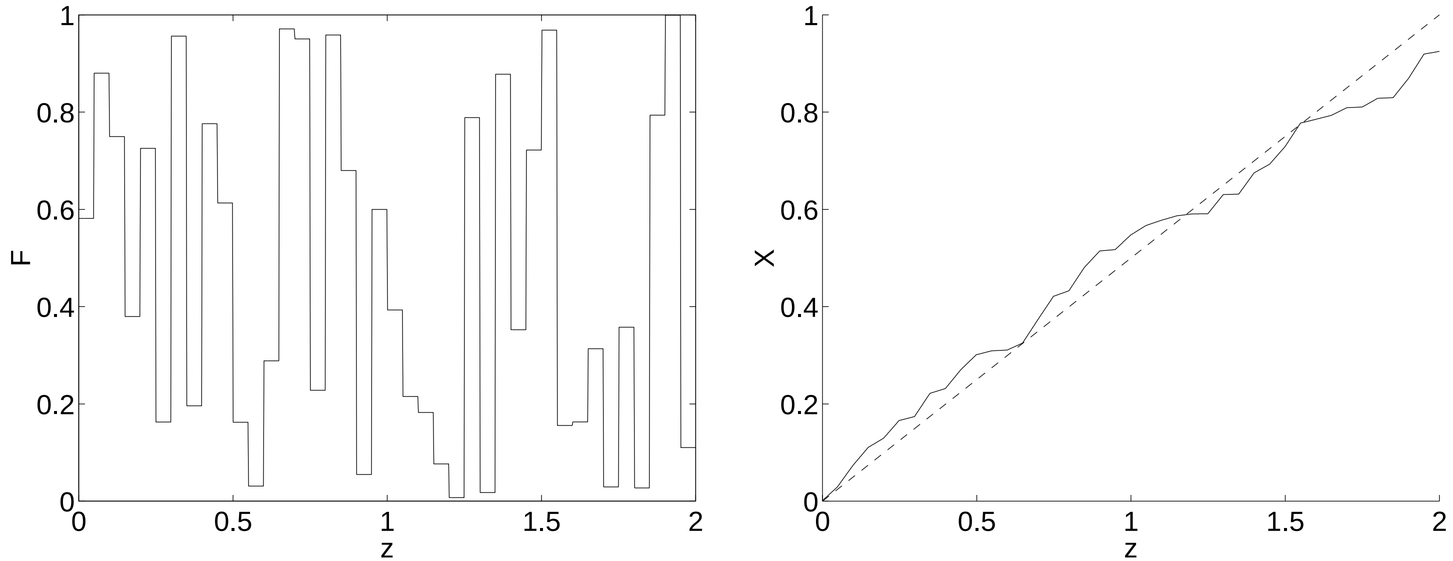


$\varepsilon = 0.2$

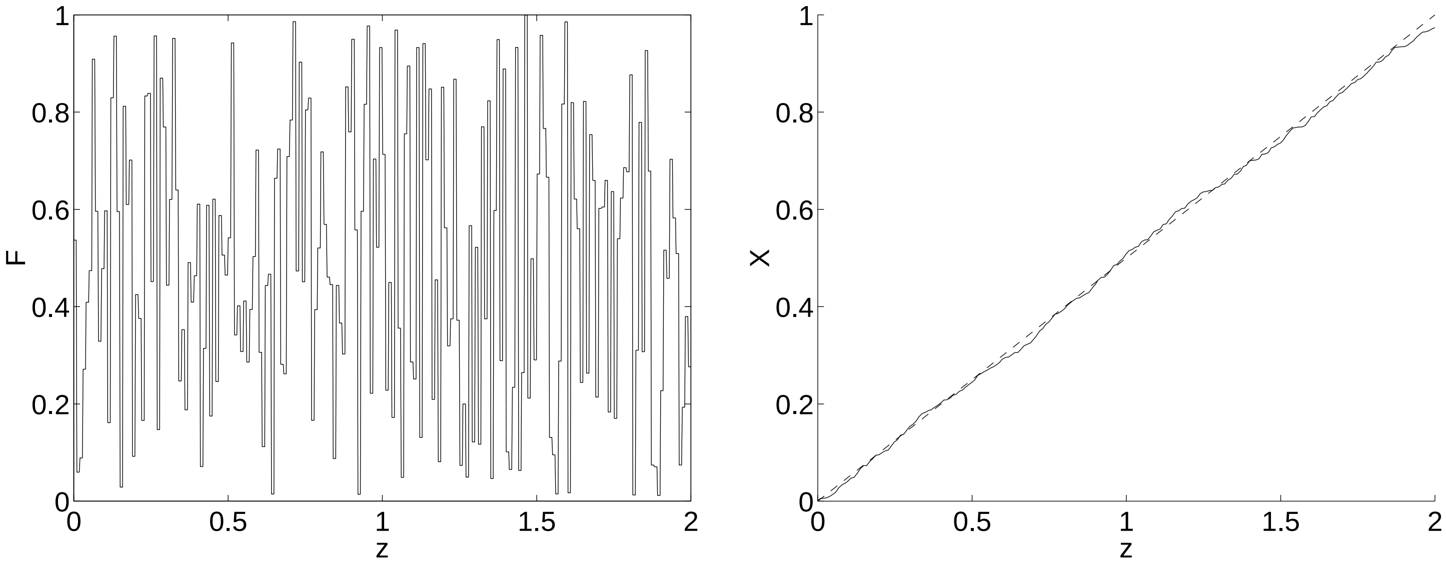
$$
\begin{aligned}
X^\varepsilon(z) \quad &= \int_0^z F\left(\frac{s}{\varepsilon}\right)ds = \varepsilon \int_0^{\frac{z}{\varepsilon}} F(s)ds = \varepsilon \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon}\right]} F_i \right) + \varepsilon \int_{\left[\frac{z}{\varepsilon}\right]}^{\frac{z}{\varepsilon}} F(s)ds \\
&= \underset{z}{\underset{\varepsilon \to 0 \downarrow}{\varepsilon \left[\frac{z}{\varepsilon}\right]}} \quad \times \quad \underset{\mathbb{E}[F_1] = \bar{F}}{\underset{\text{a.s.} \downarrow (LLN)}{\frac{1}{\left[\frac{z}{\varepsilon}\right]} \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon}\right]} F_i \right)}} \quad + \quad \underset{0}{\underset{\text{a.s.} \downarrow}{\varepsilon \left(\frac{z}{\varepsilon} - \left[\frac{z}{\varepsilon}\right] \right) F_{\left[\frac{z}{\varepsilon}\right]}}}
\end{aligned}
$$

Thus:

$$
X^\varepsilon(z) \xrightarrow{\varepsilon \to 0} \bar{X}(z), \qquad \frac{d\bar{X}}{dz} = \bar{F}.
$$



$$\varepsilon = 0.05$$



$$\varepsilon = 0.01$$

Next goal: determine the limit $\lim_{\varepsilon \to 0} X^\varepsilon(z)$ where

$$\frac{dX^\varepsilon}{dz} = F\left(\frac{z}{\varepsilon}\right)$$

for a fairly general random process $(F(z))_{z \geq 0}$.

Ergodic theory

Ergodic Theorem. If F satisfies the ergodic hypothesis, then

$$\frac{1}{Z}\int_0^Z F(z)dz \stackrel{Z\to\infty}{\longrightarrow} \bar{F} \qquad \text{a.s., where } \bar{F} = \mathbb{E}[F(0)] = \mathbb{E}[F(z)]$$

Ergodic hypothesis = "the orbit $(F(z))_{z\geq 0}$ visits all of phase space".

Ergodic theorem = "the spatial average is equivalent to the statistical average".

Counter-example for the ergodic hypothesis:

Let F_1 and F_2 be stationary, both satisfy the ergodic theorem, $\bar{F}_j = \mathbb{E}[F_j(z)]$, $j = 1, 2$, with $\bar{F}_1 \neq \bar{F}_2$.

Flip a coin (independently of F_j) $\to$ random variable $\chi = 0$ or 1 with probability $1/2$.

Let $F(z) = \chi F_1(z) + (1-\chi)F_2(z)$.

F is a stationary process with mean $\bar{F} = \frac{1}{2}(\bar{F}_1 + \bar{F}_2)$.

$$\begin{aligned}
\frac{1}{Z}\int_0^Z F(z)dz &= \chi\left(\frac{1}{Z}\int_0^Z F_1(z)dz\right) + (1-\chi)\left(\frac{1}{Z}\int_0^Z F_2(z)dz\right) \\
&\stackrel{Z\to\infty}{\longrightarrow} \chi\bar{F}_1 + (1-\chi)\bar{F}_2
\end{aligned}$$

which is a random limit different from $\bar{F}$.

The limit depends on χ because F has been trapped in a part of phase space.

Mean square theory

Let F be a stationary process, $\mathbb{E}[F(0)^2] < \infty$. Its autocorrelation function is:

$$R(z) = \mathbb{E}\left[(F(z_0) - \bar{F})(F(z_0 + z) - \bar{F})\right]$$

- R is independent of z_0 by stationarity of F.
- $|R(z)| \leq R(0)$ by Cauchy-Schwarz:

$$|R(z)| \leq \mathbb{E}\left[(F(0) - \bar{F})^2\right]^{1/2} \mathbb{E}\left[(F(z) - \bar{F})^2\right]^{1/2} = R(0)$$

- R is an even function $R(-z) = R(z)$:

$$\begin{aligned} R(-z) &= \mathbb{E}\left[(F(z_0 - z) - \bar{F})(F(z_0) - \bar{F})\right] \\ &\overset{z_0=z}{=} \mathbb{E}\left[(F(0) - \bar{F})(F(z) - \bar{F})\right] = R(z) \end{aligned}$$

Proposition. Assume $\int_0^\infty |R(z)| dz < \infty$. Let $S(Z) = \frac{1}{Z}\int_0^Z F(z) dz$. Then

$$\mathbb{E}\left[(S(Z) - \bar{F})^2\right] \overset{Z\to\infty}{\longrightarrow} 0$$

Corollary. For any $\delta > 0$

$$\mathbb{P}\left(|S(Z) - \bar{F}| > \delta\right) \leq \frac{\mathbb{E}\left[(S(Z) - \bar{F})^2\right]}{\delta^2} \overset{Z\to\infty}{\longrightarrow} 0$$

We can show that

$$Z\mathbb{E}\left[(S(Z)-\bar{F})^2\right] \xrightarrow{Z\to\infty} 2\int_0^\infty R(z)dz$$

Proof:

$$\begin{aligned}
\mathbb{E}\left[(S(Z)-\bar{F})^2\right] &= \mathbb{E}\left[\frac{1}{Z^2}\int_0^Z dz_1 \int_0^Z dz_2 (F(z_1)-\bar{F})(F(z_2)-\bar{F})\right] \\
&= \frac{2}{Z^2}\int_0^Z dz_1 \int_0^{z_1} dz_2 R(z_1-z_2) \\
&= \frac{2}{Z^2}\int_0^Z dz \int_0^{Z-z} dh R(z) \\
&= \frac{2}{Z}\int_0^Z \frac{Z-z}{Z} R(z)dz
\end{aligned}$$

Thus, denoting $R_Z(z) = \frac{Z-z}{Z}R(z)\mathbf{1}_{[0,Z]}(z)$, and using Lebesgue's theorem:

$$Z\mathbb{E}\left[(S(Z)-\bar{F})^2\right] = 2\int_0^\infty R_Z(z)dz \xrightarrow{Z\to\infty} 2\int_0^\infty R(z)dz$$

Let F be a stationary zero-mean random process. Denote

$$S_k(Z) = \frac{1}{\sqrt{Z}} \int_0^Z e^{ikz} F(z) dz$$

We can show similarly

$$\mathbb{E}[|S_k(Z)|^2] \xrightarrow{Z \to \infty} 2 \int_0^\infty R(z) \cos(kz) dz = \int_{-\infty}^\infty R(z) e^{ikz} dz$$

Simplified form of Bochner's theorem: If F is a stationary process, then the Fourier transform of its autocorrelation function is nonnegative.

Next goal: determine the limit $\lim_{\varepsilon\to 0} \boldsymbol{X}^{\varepsilon}(z)$ where

$$\frac{d\boldsymbol{X}^{\varepsilon}}{dz} = \boldsymbol{F}\left(\frac{z}{\varepsilon}, \boldsymbol{X}^{\varepsilon}(z)\right)$$

Method of averaging: Khasminskii theorem

$$\frac{d\boldsymbol{X}^{\varepsilon}}{dz} = \boldsymbol{F}(\frac{z}{\varepsilon}, \boldsymbol{X}^{\varepsilon}), \quad \boldsymbol{X}^{\varepsilon}(0) = \boldsymbol{x}_0$$

Assume:

$\boldsymbol{x} \mapsto \boldsymbol{F}(z, \boldsymbol{x})$ is Lipschitz,

$z \mapsto \boldsymbol{F}(z, \boldsymbol{x})$ is stationary and ergodic (or with integrable correlation function).

Define:

$$\bar{\boldsymbol{F}}(\boldsymbol{x}) = \mathbb{E}[\boldsymbol{F}(z, \boldsymbol{x})]$$

Let $\bar{\boldsymbol{X}}$ be the solution of

$$\frac{d\bar{\boldsymbol{X}}}{dz} = \bar{\boldsymbol{F}}(\bar{\boldsymbol{X}}), \quad \bar{\boldsymbol{X}}(0) = \boldsymbol{x}_0$$

Theorem: for any $Z > 0$,

$$\sup_{z \in [0, Z]} \mathbb{E}\left[|\boldsymbol{X}^{\varepsilon}(z) - \bar{\boldsymbol{X}}(z)|\right] \stackrel{\varepsilon \to 0}{\longrightarrow} 0$$

[1] R. Z. Khasminskii, Theory Probab. Appl. **11** (1966), 211-228.

Application to wave propagation in random media

Equations for the Fourier components of the wave:

$$\frac{d\boldsymbol{X}^\varepsilon}{dz} = \boldsymbol{F}\big(\frac{z}{\varepsilon}, \boldsymbol{X}^\varepsilon\big),$$

where

$$\boldsymbol{X}^\varepsilon = \begin{pmatrix} \hat{p} \\ \hat{u} \end{pmatrix}, \qquad \boldsymbol{F}(z, \boldsymbol{X}) = -i\omega \begin{pmatrix} 0 & \rho(z) \\ \frac{1}{\kappa(z)} & 0 \end{pmatrix} \boldsymbol{X}$$

Apply the method of averaging $\Longrightarrow$ $X^\varepsilon(\omega, z)$ converges in $L^1(\mathbb{P})$ to $\bar{X}(\omega, z)$ solution of

$$\frac{d\bar{\boldsymbol{X}}}{dz} = -i\omega \begin{pmatrix} 0 & \bar{\rho} \\ \frac{1}{\bar{\kappa}} & 0 \end{pmatrix} \bar{\boldsymbol{X}}, \qquad \bar{\rho} = \mathbb{E}[\rho], \qquad \bar{\kappa} = \big(\mathbb{E}[\kappa^{-1}]\big)^{-1}$$

$\hookrightarrow$ deterministic "effective medium" with parameters $\bar{\rho}$, $\bar{\kappa}$.

Let $(\bar{p}, \bar{u})$ be the solution of the homogeneous effective system

$$\bar{\rho}\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{p}}{\partial z} = 0$$
$$\frac{\partial \bar{p}}{\partial t} + \bar{\kappa}\frac{\partial \bar{u}}{\partial z} = 0$$

The propagation speed of $(\bar{p}, \bar{u})$ is $\bar{c} = \sqrt{\bar{\kappa}/\bar{\rho}}$.

Compare $p^\varepsilon(t,z)$ with $\bar{p}(t,z)$:

$$\begin{aligned} \mathbb{E}\left[|p^\varepsilon(t,z) - \bar{p}(t,z)|\right] &= \frac{1}{2\pi}\mathbb{E}\left[\left|\int e^{-i\omega t}(\hat{p}^\varepsilon(\omega,z) - \hat{\bar{p}}(\omega,z))d\omega\right|\right] \\ &\leq \frac{1}{2\pi}\int \mathbb{E}\left[|\hat{p}^\varepsilon(\omega,z) - \hat{\bar{p}}(\omega,z)|\right]d\omega \end{aligned}$$

The dominated convergence theorem then gives the convergence in $L^1(\mathbb{P})$ of p^ε to $\bar{p}$ in the time domain.
$\hookrightarrow$ the effective speed of the acoustic wave $(p^\varepsilon, u^\varepsilon)$ as $\varepsilon \to 0$ is $\bar{c}$.

This analysis is just a small piece of the homogenization theory.
cf book *The theory of composites* by G. Milton.

Example: bubbles in water

$\rho_a = 1.2\ 10^3$ g/m^3, $\kappa_a = 1.4\ 10^8$ g/s^2/m, $c_a = 340$ m/s.
$\rho_w = 1.0\ 10^6$ g/m^3, $\kappa_w = 2.0\ 10^{12}$ g/s^2/m, $c_w = 1425$ m/s.
If the typical pulse frequency is 10 Hz - 30 kHz, then the typical wavelength is 1 cm - 100 m. The bubble sizes are much smaller $\Longrightarrow$ the effective medium theory can be applied.

$$\bar{\rho} = \mathbb{E}[\rho] = \phi\rho_a + (1-\phi)\rho_w = \begin{cases} 9.9\ 10^5\ \text{g/m}^3 & \text{if } \phi = 1\% \\ 9\ 10^5\ \text{g/m}^3 & \text{if } \phi = 10\% \end{cases}$$

$$\bar{\kappa} = \left(\mathbb{E}[\kappa^{-1}]\right)^{-1} = \left(\frac{\phi}{\kappa_a} + \frac{1-\phi}{\kappa_w}\right)^{-1} = \begin{cases} 1.4\ 10^{10}\ \text{g/s}^2/\text{m} & \text{if } \phi = 1\% \\ 1.4\ 10^9\ \text{g/s}^2/\text{m} & \text{if } \phi = 10\% \end{cases}$$

where $\phi =$ volume fraction of air.
Thus, $\bar{c} = 120$ m/s if $\phi = 1\%$ and $\bar{c} = 37$ m/s if $\phi = 10$ %.

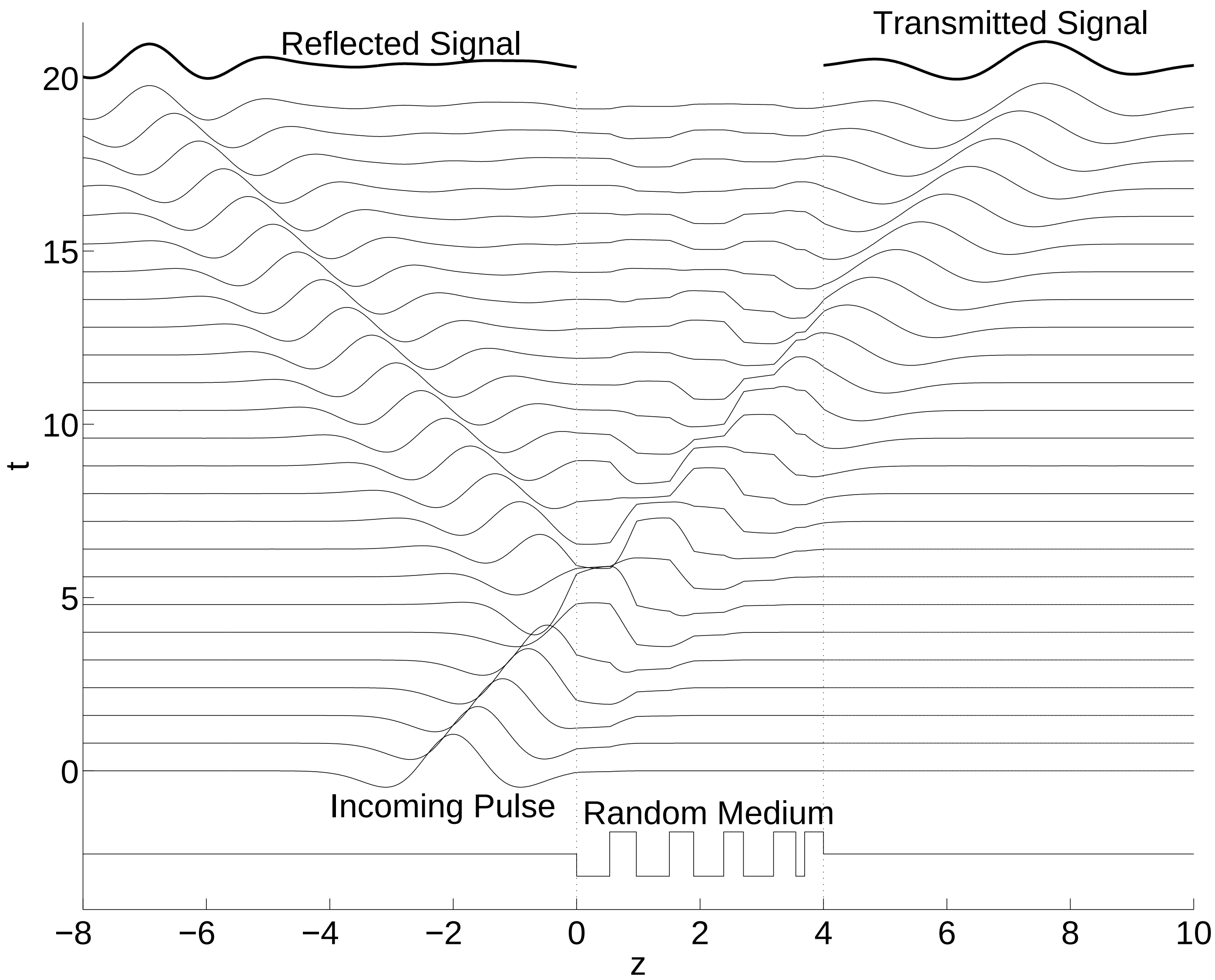
$\hookrightarrow$ the average sound speed $\bar{c}$ can be much smaller than ess inf(c).

The converse is impossible:

$$\mathbb{E}[c^{-1}] = \mathbb{E}\left[\kappa^{-1/2}\rho^{1/2}\right] \leq \mathbb{E}[\kappa^{-1}]^{1/2}\mathbb{E}[\rho]^{1/2} = \bar{c}^{-1}$$

Thus $\bar{c} \leq \mathbb{E}[c^{-1}]^{-1} \leq$ ess sup(c).

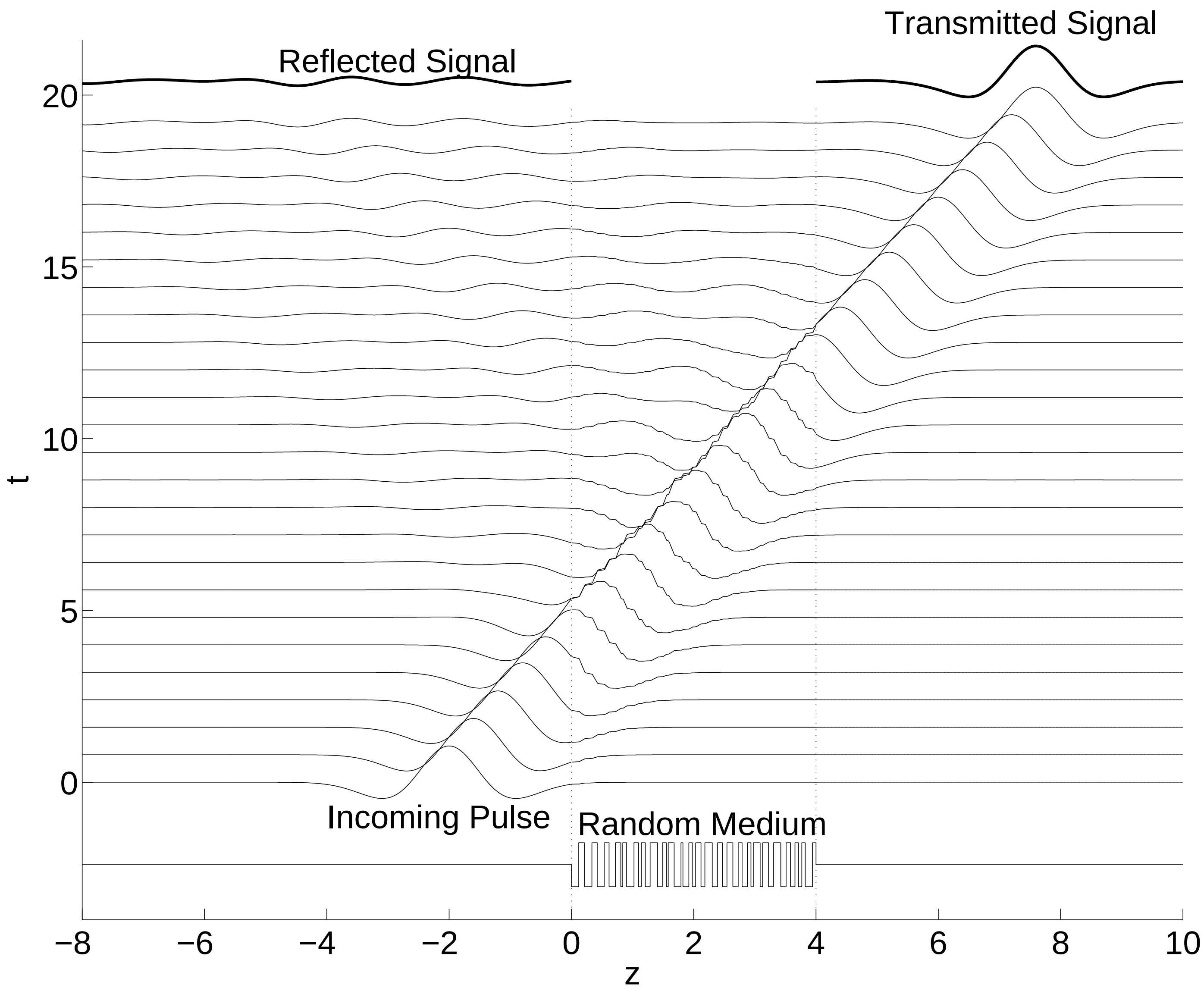
Propagation through a stack of random layers



Sizes of the layers: i.i.d. with uniform distribution over $[0.2, 0.6]$ (mean 0.4).

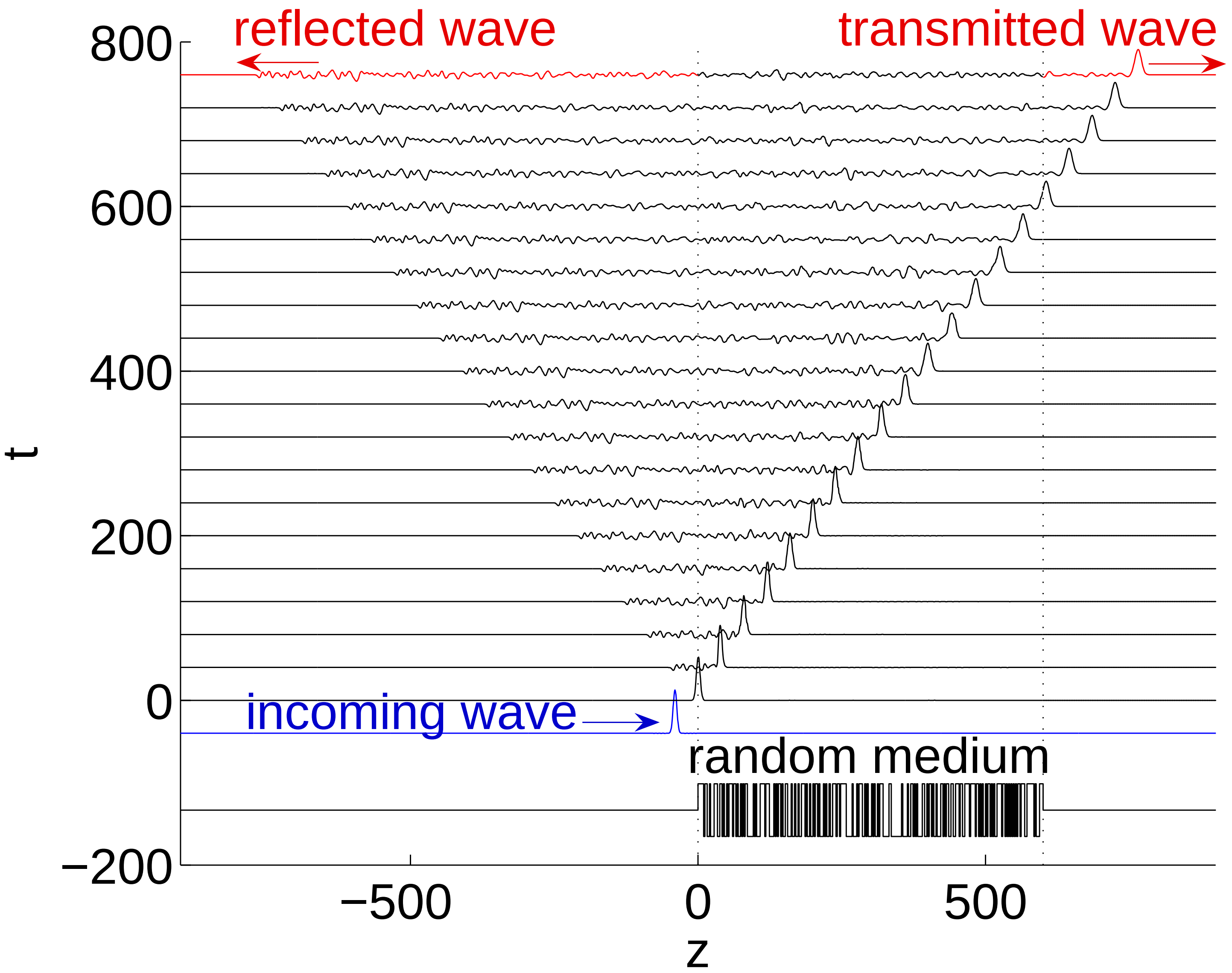
Medium parameters $\rho \equiv 1$, $1/\kappa_a = 0.2$, $1/\kappa_b = 1.8$.

Propagation through a stack of random layers



Sizes of the layers: i.i.d. with uniform distribution over $[0.04, 0.12]$ (mean 0.08).

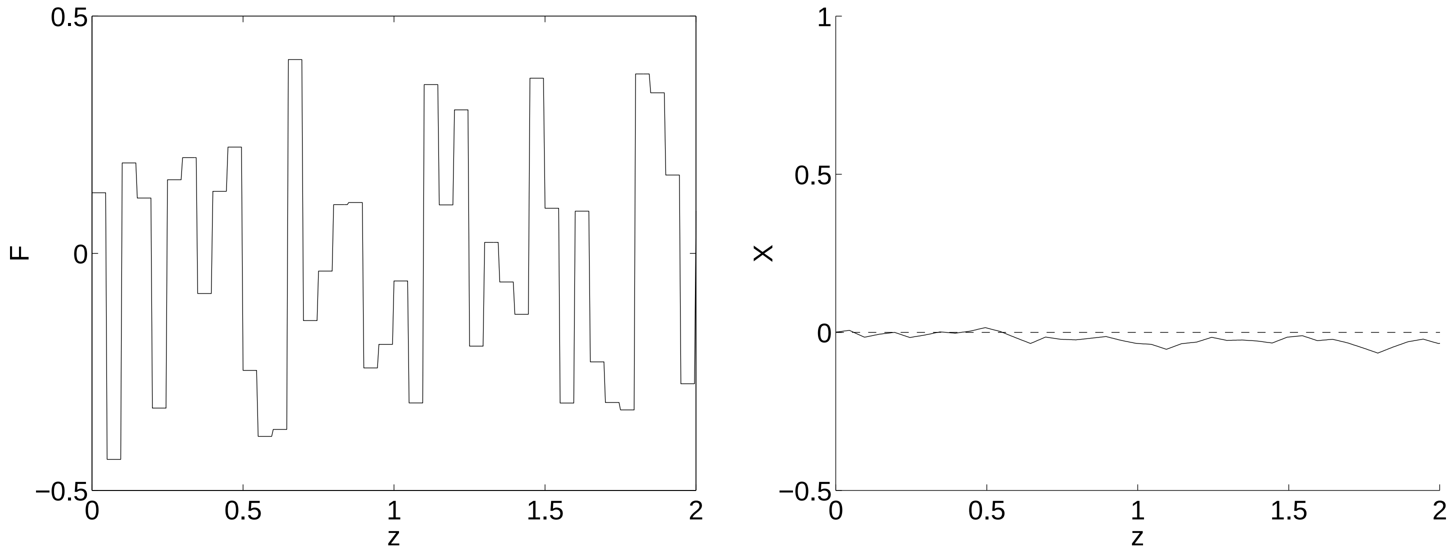
Long distance propagation



Toy model

$$\frac{dX^\varepsilon}{dz} = F\left(\frac{z}{\varepsilon}\right)$$

with $F(z) = \sum_{i=1}^{\infty} F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = \bar{F} = 0$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$.



$\varepsilon = 0.05$

For any $z \in [0, Z]$, we have

$$X^\varepsilon(z) \xrightarrow{\varepsilon \to 0} \bar{X}(z), \qquad \frac{d\bar{X}}{dz} = \bar{F} = 0.$$

No macroscopic evolution is noticeable.

$\rightarrow$ it is necessary to look at larger z to get an effective behavior

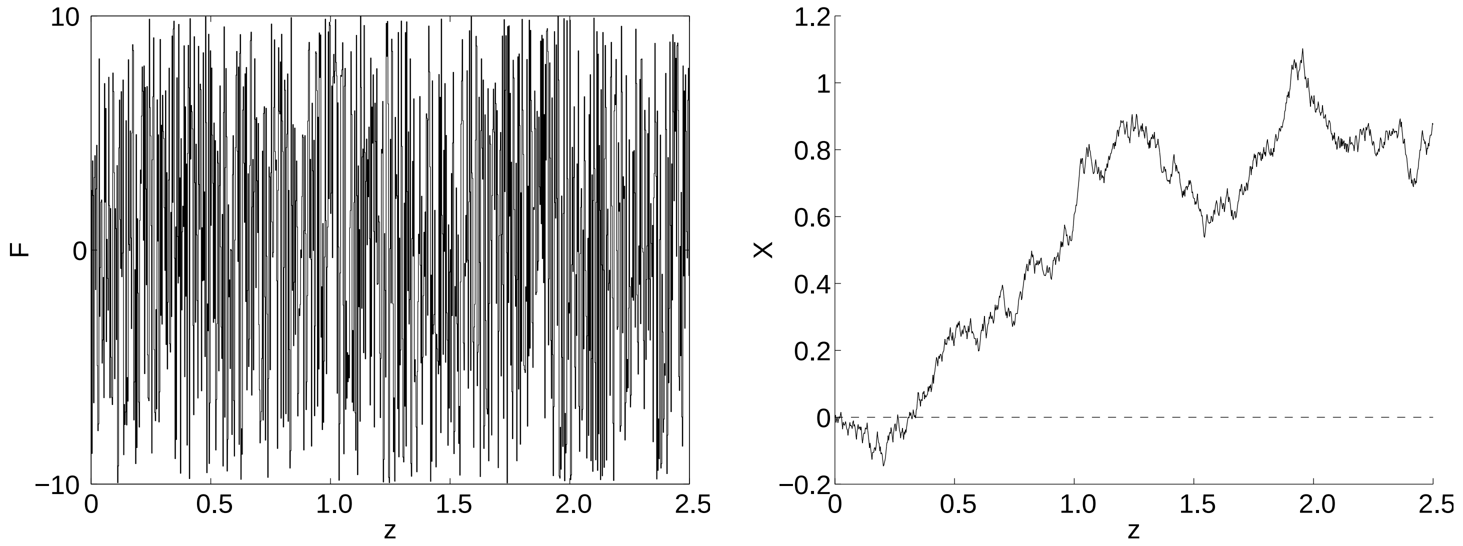
$$z \mapsto \frac{z}{\varepsilon}, \qquad \tilde{X}^\varepsilon(z) = X^\varepsilon(\frac{z}{\varepsilon})$$

$$\frac{d\tilde{X}^\varepsilon}{dz} = \frac{1}{\varepsilon} F(\frac{z}{\varepsilon^2})$$

Diffusion-approximation: Toy model

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon} F\left(\frac{z}{\varepsilon^2}\right)$$

with $F(z) = \sum_{i=1}^{\infty} F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = 0$ and $\mathbb{E}[F_i^2] = \sigma^2$.



$\varepsilon = 0.05$

$$
\begin{aligned}
X^\varepsilon(z) &= \int_0^z \frac{1}{\varepsilon} F\Big(\frac{s}{\varepsilon^2}\Big) ds = \varepsilon \int_0^{\frac{z}{\varepsilon^2}} F(s) ds = \varepsilon \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon^2}\right]} F_i \right) + \varepsilon \int_{\left[\frac{z}{\varepsilon^2}\right]}^{\frac{z}{\varepsilon^2}} F(s) ds \\
&= \underbrace{\varepsilon \sqrt{\left[\frac{z}{\varepsilon^2}\right]}}_{\substack{\varepsilon \to 0 \downarrow \\ \sqrt{z}}} \times \underbrace{\frac{1}{\sqrt{\left[\frac{z}{\varepsilon^2}\right]}} \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon^2}\right]} F_i \right)}_{\substack{\text{law} \downarrow (CLT) \\ \mathcal{N}(0,\sigma^2)}} + \underbrace{\varepsilon \left(\frac{z}{\varepsilon^2} - \left[\frac{z}{\varepsilon^2}\right]\right) F_{\left[\frac{z}{\varepsilon^2}\right]}}_{\substack{\text{a.s.} \downarrow \\ 0}}
\end{aligned}
$$

Thus: $X^\varepsilon(z)$ converges in distribution as $\varepsilon \to 0$ to $\bar{X}(z)$ whose distribution is $\mathcal{N}(0, \sigma^2 z)$. Its pdf is

$$
p_z(\bar{x}) = \frac{1}{\sqrt{2\pi\sigma^2 z}} \exp\Big(- \frac{\bar{x}^2}{2\sigma^2 z} \Big)
$$

With some more work: The process $(X^\varepsilon(z))_{z\geq 0}$ converges in distribution to a Brownian motion $(\sigma W_z)_{z\geq 0}$.

(a Brownian motion is a Gaussian process with mean zero and covariance function $\mathbb{E}[W_z W_{z'}] = \min(z, z')$).

The pdf $p_z(x)$ of the limit process $\bar{X}(z) = \sigma W_z$ satisfies

$$
\frac{\partial p_z}{\partial z} = \frac{\sigma^2}{2} \frac{\partial^2 p_z}{\partial \bar{x}^2}, \qquad p_{z=0}(\bar{x}) = \delta(\bar{x})
$$

Next goal: determine the limit $\lim_{\varepsilon\to 0} X^\varepsilon(z)$ where

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon} F\left(\frac{z}{\varepsilon^2}\right)$$

for a fairly general process $(F(z))_{z\geq 0}$ with $\mathbb{E}[F(z)] = 0$.

Write

$$X^{\varepsilon}(z) = \frac{1}{\varepsilon}\int_0^z F\left(\frac{s}{\varepsilon^2}\right)ds = \varepsilon\sqrt{\frac{z}{\varepsilon^2}} \times \frac{1}{\sqrt{\frac{z}{\varepsilon^2}}}\int_0^{\frac{z}{\varepsilon^2}} F(s)\,ds$$

We know that

$$\begin{aligned}
\mathbb{E}\Big[\frac{1}{\sqrt{\frac{z}{\varepsilon^2}}}\int_0^{\frac{z}{\varepsilon^2}} F(s)\,ds\Big] &= 0, \\
\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big(\frac{1}{\sqrt{\frac{z}{\varepsilon^2}}}\int_0^{\frac{z}{\varepsilon^2}} F(s)\,ds\Big)^2\Big] &= \lim_{Z\to\infty} Z\mathbb{E}\big[S(Z)^2\big], \qquad S(Z) = \frac{1}{Z}\int_0^Z F(s)\,ds \\
&= 2\int_0^{\infty}\mathbb{E}[F(0)F(u)]du,
\end{aligned}$$

The gaussian property of the limit of $X^{\varepsilon}(z)$ is ensured by an invariance principle.

Conclusion:

$$\frac{1}{\varepsilon}\int_0^z F\Big(\frac{s}{\varepsilon^2}\Big)ds \stackrel{\varepsilon\to 0}{\longrightarrow} \sqrt{2}\sigma W_z$$

in distribution, where $(W_z)_{z\geq 0}$ is a Brownian motion and

$$\sigma^2 = \int_0^{\infty}\mathbb{E}[F(0)F(u)]du$$

Next goal: determine the limit $\lim_{\varepsilon \to 0} \boldsymbol{X}^{\varepsilon}(z)$ where

$$\frac{d\boldsymbol{X}^{\varepsilon}}{dz} = \frac{1}{\varepsilon}\boldsymbol{F}\left(\frac{z}{\varepsilon^2}, \boldsymbol{X}^{\varepsilon}(z)\right)$$

when

$$\mathbb{E}[\boldsymbol{F}(z, \boldsymbol{x})] = 0 \quad \forall \boldsymbol{x}$$

Diffusion-approximation

$$\frac{d\boldsymbol{X}^\varepsilon}{dz}(z) = \frac{1}{\varepsilon}\boldsymbol{F}\Big(\frac{z}{\varepsilon^2}, \boldsymbol{X}^\varepsilon(z)\Big), \qquad \boldsymbol{X}^\varepsilon(0) = \boldsymbol{x}_0 \in \mathbb{R}^d.$$

$\boldsymbol{F}$ stationary and ergodic (in z) centered: $\mathbb{E}[\boldsymbol{F}(z,\boldsymbol{x})] = \boldsymbol{0}$.

Theorem: *The processes $(\boldsymbol{X}^\varepsilon(z))_{z\geq 0}$ converge in distribution in $\mathcal{C}^0([0,\infty),\mathbb{R}^d)$ to the diffusion (Markov) process $\boldsymbol{X}$ with the generator $\mathcal{L}$:*

$$\mathcal{L} = \sum_{i,j=1}^d a_{ij}(\boldsymbol{x})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(\boldsymbol{x})\frac{\partial}{\partial x_j}$$

with

$$\begin{aligned} a_{ij}(\boldsymbol{x}) &= \int_0^\infty \mathbb{E}\left[F_i(0,\boldsymbol{x})F_j(u,\boldsymbol{x})\right] du \\ b_j(\boldsymbol{x}) &= \sum_{i=1}^d \int_0^\infty \mathbb{E}\left[F_i(0,\boldsymbol{x})\partial_{x_i}F_j(u,\boldsymbol{x})\right] du \end{aligned}$$

It means that the pdf $p_z(\boldsymbol{x})$ of $\boldsymbol{X}(z)$ satisfies:

$$\frac{\partial p_z}{\partial z} = \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j}\big(a_{ij}(\boldsymbol{x})p_z\big) - \sum_{j=1}^d \frac{\partial}{\partial x_j}\big(b_j(\boldsymbol{x})p_z\big)$$

Diffusion-approximation - the linear case

$$\frac{d\boldsymbol{X}^\varepsilon}{dz}(z) = \frac{1}{\varepsilon}\mathbf{F}\left(\frac{z}{\varepsilon^2}\right)\boldsymbol{X}^\varepsilon(z), \qquad \boldsymbol{X}^\varepsilon(0) = \boldsymbol{x}_0 \in \mathbb{R}^d.$$

$\mathbf{F}$ matrix-valued, stationary, ergodic, and centered: $\mathbb{E}[\mathbf{F}(z)] = \mathbf{0}$.

Theorem: *The expectations* $(\mathbb{E}[\boldsymbol{X}^\varepsilon(z)])_{z\geq 0}$ *converge in* $\mathcal{C}^0([0,\infty),\mathbb{R}^d)$ *to the solution* $(\bar{\boldsymbol{X}}(z))_{z\geq 0}$ *of the linear system:*

$$\frac{d\bar{\boldsymbol{X}}}{dz}(z) = \bar{\mathbf{F}}\bar{\boldsymbol{X}}(z), \qquad \bar{\boldsymbol{X}}(0) = \boldsymbol{x}_0 \in \mathbb{R}^d.$$

with

$$\bar{F}_{jl} = \sum_{k=1}^{d}\int_0^\infty \mathbb{E}\big[F_{kl}(0)F_{jk}(u)\big]du$$

Diffusion-approximation - a simple example

$$\frac{dX^\varepsilon}{dz} = \frac{i}{\varepsilon}\mu\big(\frac{z}{\varepsilon^2}\big)X^\varepsilon, \qquad X^\varepsilon(z=0) = x_0$$

Method 1. Apply the theorem to

$$\frac{d}{dz}\begin{pmatrix} X_r^\varepsilon \\ X_i^\varepsilon \end{pmatrix} = \frac{1}{\varepsilon}\mu\big(\frac{z}{\varepsilon^2}\big)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} X_r^\varepsilon \\ X_i^\varepsilon \end{pmatrix}$$

The expectation converges to

$$\frac{d}{dz}\begin{pmatrix} \bar{X}_r \\ \bar{X}_i \end{pmatrix} = \sigma^2\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \bar{X}_r \\ \bar{X}_i \end{pmatrix}$$

where

$$\sigma^2 = \int_0^\infty \mathbb{E}[\mu(0)\mu(u)]du$$

This shows that

$$\mathbb{E}\big[X^\varepsilon(z)\big] \xrightarrow{\varepsilon\to 0} \bar{X}(z)$$

where $\bar{X}(z)$ is the solution of

$$\frac{d\bar{X}}{dz} = -\sigma^2\bar{X}, \qquad \bar{X}(z=0) = x_0$$

Method 2. We have

$$X^\varepsilon(z) = x_0 \exp\Big(\frac{i}{\varepsilon}\int_0^z \mu(\frac{s}{\varepsilon^2})ds\Big)$$

We have

$$\frac{1}{\varepsilon}\int_0^z \mu(\frac{s}{\varepsilon^2})ds \overset{\varepsilon\to 0}{\longrightarrow} \sqrt{2}\sigma W_z$$

in distribution, where $(W_z)_{z\geq 0}$ is a Brownian motion and

$$\sigma^2 = \int_0^\infty \mathbb{E}[\mu(0)\mu(u)]du$$

Therefore

$$X^\varepsilon(z) \overset{\varepsilon\to 0}{\longrightarrow} X(z) := \exp\big(i\sqrt{2}\sigma W_z\big)$$

in distribution. Note

$$\mathbb{E}\big[\exp\big(i\sqrt{2}\sigma W_z\big)\big] = \frac{1}{\sqrt{2\pi z}}\int_{-\infty}^\infty e^{i\sqrt{2}\sigma w}e^{-\frac{w^2}{2z}}dw = e^{-\frac{\sigma^2 z}{2}}$$

This shows that

$$\mathbb{E}\big[X^\varepsilon(z)\big] \overset{\varepsilon\to 0}{\longrightarrow} \bar{X}(z) = x_0 e^{-\sigma^2 z}$$

where $\bar{X}(z)$ is the solution of

$$\frac{d\bar{X}}{dz} = -\sigma^2\bar{X}, \qquad \bar{X}(z=0) = x_0$$

We have shown that the expectation of the solution of

$$\frac{dX^\varepsilon}{dz} = \frac{i}{\varepsilon}\mu\Big(\frac{z}{\varepsilon^2}\Big)X^\varepsilon, \qquad X^\varepsilon(z=0) = x_0$$

converges as $\varepsilon \to 0$ to the solution of

$$\frac{d\bar{X}}{dz} = -\sigma^2 \bar{X}, \qquad \bar{X}(z=0) = x_0$$

$\hookrightarrow$ random phases give rise to effective damping.

Wave propagation in random media: the random paraxial regime

- Propagation along a privileged axis (i.e. laser beam, ...).

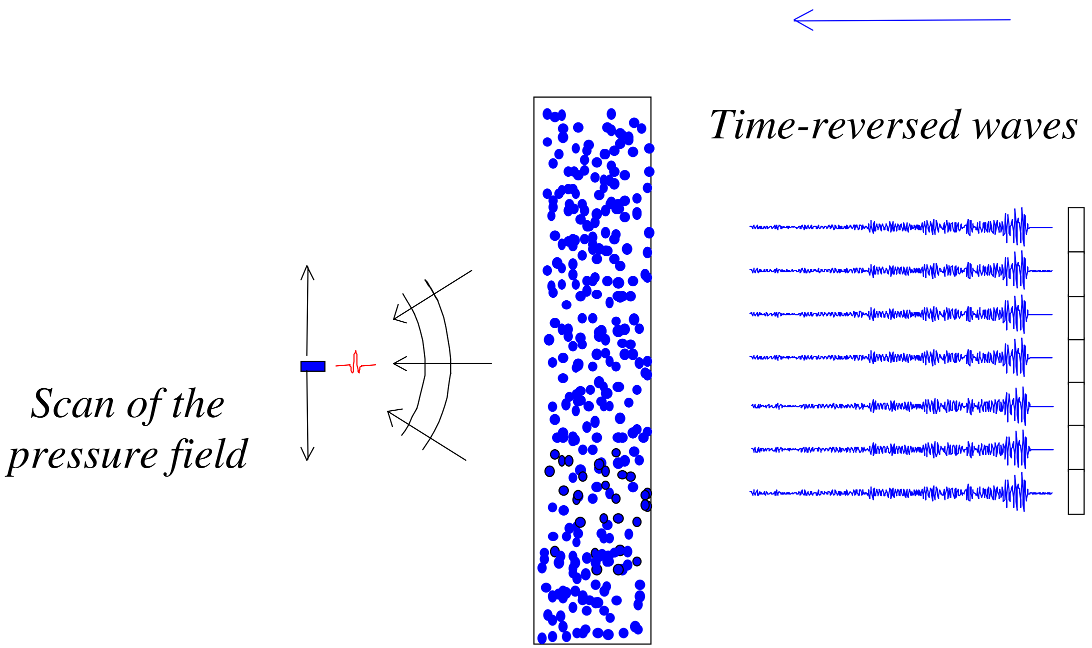
- Experiments: time-reversal experiments in ultrasound acoustics.

Experimental observations:
- robust spatial refocusing
- focal spot smaller in random medium than in homogeneous medium

- Analysis of the mechanisms responsible for statistically stable time reversal and enhanced refocusing in random media.

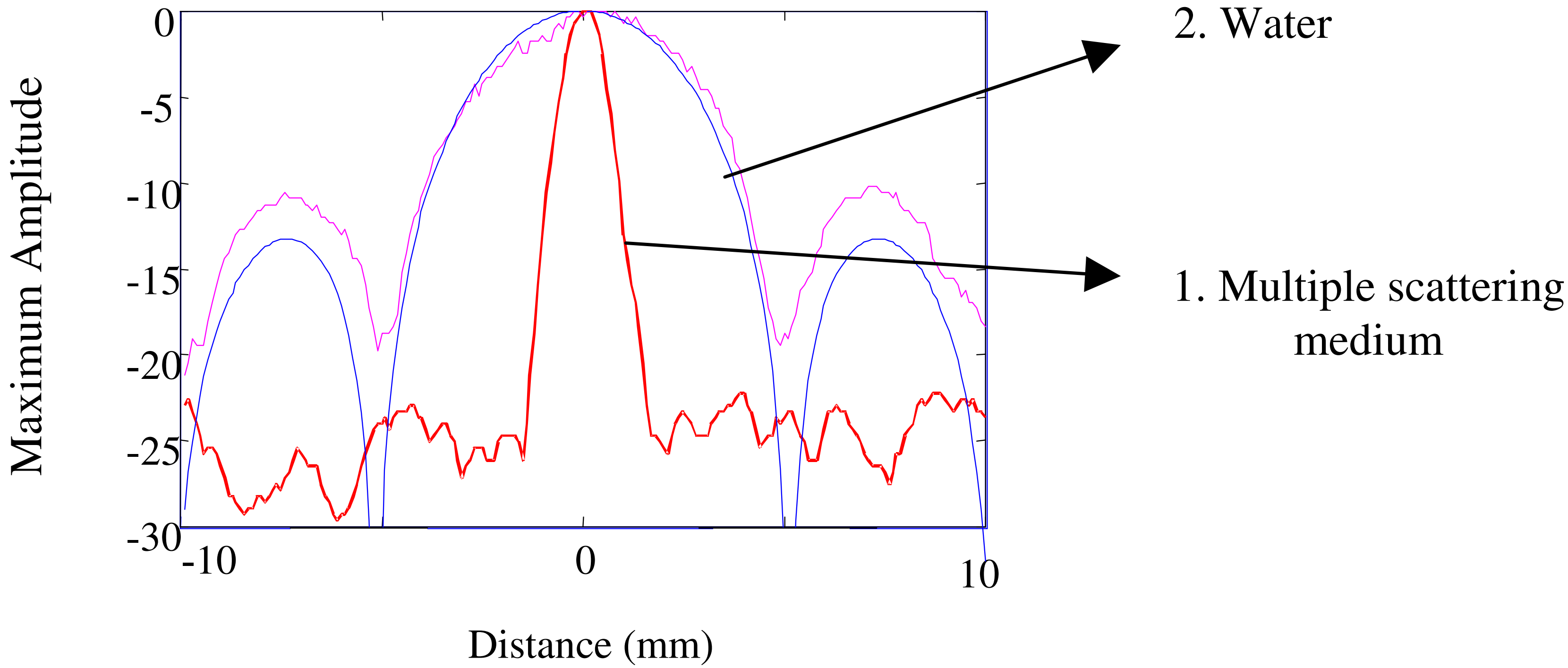
Statistical analysis of $\overline{\hat{G}}\hat{G}$ in the paraxial regime.

(*a*)



Time-reversal experiment

(*b*)



Physical derivation of the paraxial wave equation

Wave equation in homogeneous medium:

$$(\partial_z^2 + \Delta_\perp)u - \frac{1}{c_0^2}\partial_t^2 u = 0$$

Fourier transform

$$\hat{u}(\omega, z, \boldsymbol{x}) = \int u(t, z, \boldsymbol{x}) e^{i\omega t} dt$$

$\hookrightarrow$ Helmholtz equation in homogeneous medium:

$$(\partial_z^2 + \Delta_\perp)\hat{u} + k^2 \hat{u} = 0$$

with initial/incoming and boundary/radiation conditions. Here $k = \omega / c_0$.

A particular solution: plane wave going into the z direction $\hat{u} = e^{ikz}$.

Ansatz: slowly-varying envelope around a plane wave going into the z direction

$$\hat{u}(\omega, z, \boldsymbol{x}) = e^{ikz}\phi(\omega, z, \boldsymbol{x})$$

$$\hookrightarrow \partial_z^2\phi + 2ik\partial_z\phi + \Delta_\perp\phi = 0$$

Forward-scattering approximation in direction z ($\leftrightarrow$ ϕ slowly varying in z):

$$\cancel{\partial_z^2\phi} + 2ik\partial_z\phi + \Delta_\perp\phi = 0$$

with initial condition $\phi(\omega, z = 0, \boldsymbol{x}) = \phi_0(\omega, \boldsymbol{x})$.

Mathematical derivation: scaling assumptions

$$(\partial_z^2 + \Delta_\perp)\hat{u} + k^2\big(1 + \mu(z, \boldsymbol{x})\big)\hat{u} = 0$$

1) the correlation length of the medium l_K is smaller than the propagation distance L:

$$\varepsilon^2 = \frac{l_K}{L}$$

2) the transverse width of the source R_0 and the correlation length of the medium l_K are of the same order:

$$\frac{R_0}{L} \sim \varepsilon^2$$

3) the wavelength λ_0 is smaller than the propagation distance (high-frequency regime):

$$\frac{\lambda_0}{L} \sim \varepsilon^4$$

Motivation: In the absence of random fluctuations, if one wants diffraction effects of order one for the propagation distance L:

$$\underbrace{2ik\partial_z\phi}_{\sim \frac{1}{\lambda_0 L}} \quad + \quad \underbrace{\Delta_\perp\phi}_{\sim \frac{1}{R_0^2}} \quad = 0$$

then one must have $L\lambda_0 \sim R_0^2 \Longrightarrow L\lambda_0 \sim L^2\varepsilon^4 \Longrightarrow \lambda_0/L \sim \varepsilon^4$.

Mathematical derivation: sketch of proof

$$(\partial_z^2 + \Delta_\perp)\hat{u} + k^2\big(1 + \mu(z, \boldsymbol{x})\big)\hat{u} = 0$$

Initial condition: source at $z = 0$ with high carrier frequency (small wavelength $\sim \varepsilon^4$) and lateral extent $\sim \varepsilon^2$.

Scaled regime:

$$\omega \to \frac{\omega}{\varepsilon^4}, \quad k \to \frac{k}{\varepsilon^4}, \quad \mu(z, \boldsymbol{x}) \to \varepsilon^3 \mu\Big(\frac{z}{\varepsilon^2}, \frac{\boldsymbol{x}}{\varepsilon^2}\Big)$$

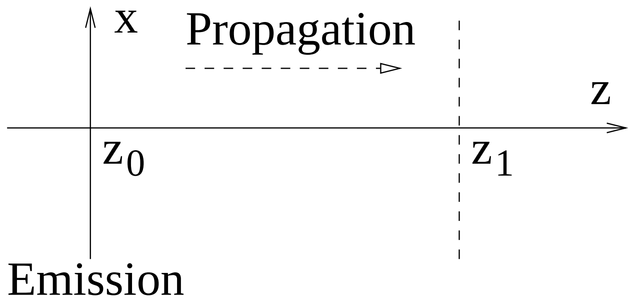
$$\hat{u}(\omega, z, \boldsymbol{x}) = e^{i\frac{k}{\varepsilon^4}z} \phi\Big(\frac{\omega}{\varepsilon^4}, z, \frac{\boldsymbol{x}}{\varepsilon^2}\Big)$$

$$\partial_z^2 \phi + \frac{1}{\varepsilon^4}\left(2ik\partial_z \phi + \Delta_\perp \phi + \frac{k^2}{\varepsilon}\mu\Big(\frac{z}{\varepsilon^2}, \frac{\boldsymbol{x}}{\varepsilon^2}\Big)\phi\right) = 0$$

In the regime $\varepsilon \ll 1$, forward-scattering approximation in direction z:

$$2ik\partial_z \phi + \Delta_\perp \phi + \frac{k^2}{\varepsilon}\mu\Big(\frac{z}{\varepsilon^2}, \frac{\boldsymbol{x}}{\varepsilon^2}\Big)\phi = 0$$

Paraxial wave propagation



- Propagation direction z. Initial condition located in the plane $z = z_0$.
- Inhomogeneous medium with local celerity $c(z, \boldsymbol{x}) = \dfrac{c_0}{n(z, \boldsymbol{x})}$.
- The field $u(t, z, \boldsymbol{x})$, $t \in \mathbb{R}$, $(z, \boldsymbol{x}) \in \mathbb{R}^3$, reads as:

$$u(t, z, \boldsymbol{x}) = \frac{1}{2\pi} \int e^{-i\omega t} \hat{u}(\omega, z, \boldsymbol{x}) d\omega$$

with $\hat{u}(\omega, z, \boldsymbol{x}) = e^{ikz} \phi(\omega, z, \boldsymbol{x})$, where $k = \omega / c_0$ and ϕ is solution of:

$$2ik\phi_z + \Delta_\perp \phi + k^2 (n^2(z, \boldsymbol{x}) - 1)\phi = 0, \qquad \phi(\omega, z = z_0, \boldsymbol{x}) = \phi_0(\omega, \boldsymbol{x})$$

$\Delta_\perp$ =transverse Laplacian (w.r.t. $\boldsymbol{x}$).

- From now on we set $\mu = n^2 - 1$.

μ is a stationary, zero-mean, ergodic process.

Green's function

- Fundamental solution $\hat{G}\big(\omega,(z,\boldsymbol{x}),(z_0,\boldsymbol{x}_0)\big)$:

$$2ik\partial_z\hat{G}+\Delta_\perp\hat{G}+k^2\mu(z,\boldsymbol{x})\hat{G}=0, \qquad (k=\omega/c_0)$$

starting from $\hat{G}\big(\omega,(z=z_0,\boldsymbol{x}),(z_0,\boldsymbol{x}_0)\big)=\delta(\boldsymbol{x}-\boldsymbol{x}_0)$.

- Solution ϕ of

$$2ik\partial_z\phi+\Delta_\perp\phi+k^2\mu(z,\boldsymbol{x})\phi=0,$$

starting from $\phi(\omega,z=0,\boldsymbol{x})=\phi_0(\omega,\boldsymbol{x})$ at $z=0$:

$$\phi(\omega,z,\boldsymbol{x})=\int\hat{G}\big(\omega,(z,\boldsymbol{x}),(0,\boldsymbol{x}_0)\big)\phi_0(\omega,\boldsymbol{x}_0)d\boldsymbol{x}_0$$

- In homogeneous media ($\mu\equiv 0$) :

$$\hat{G}\big(\omega,(z,\boldsymbol{x}),(z_0,\boldsymbol{x}_0)\big)=\frac{\exp\left(\frac{ik|\boldsymbol{x}-\boldsymbol{x}_0|^2}{2|z-z_0|}\right)}{2i\pi\frac{|z-z_0|}{k}}$$

$\hat{G}$ can be computed from the (Fourier transform of the) Schrödinger equation or from an expansion of the Green's function of the Helmholtz equation.

Transmitted wave - homogeneous case

- Solve by Fourier transform (in space):

$$\begin{cases} 2ik\partial_z\phi + \Delta_\perp\phi = 0, \\ \phi(\omega, z=0, \boldsymbol{x}) = \phi_0(\boldsymbol{x}), \end{cases}$$

$$\begin{array}{ccc} \phi(z=0,\boldsymbol{x}) & \xrightarrow{\text{DFT}} & \check{\phi}(z=0,\boldsymbol{\kappa}) \\ \text{SL}\downarrow & & \downarrow \partial_z\check{\phi} = -\frac{i|\boldsymbol{\kappa}|^2}{2k}\check{\phi} \\ \phi(z,\boldsymbol{x}) & \xleftarrow{\text{IFT}} & \check{\phi}(z,\boldsymbol{\kappa}) \end{array}$$

$$\begin{aligned} \phi(\omega,z,\boldsymbol{x}) &= \frac{1}{(2\pi)^{d-1}}\int \check{\phi}_0(\omega,\boldsymbol{\kappa}) \exp i\left(\boldsymbol{\kappa}\cdot\boldsymbol{x} - \frac{|\boldsymbol{\kappa}|^2}{2k}z\right) d\boldsymbol{\kappa}, \\ \check{\phi}_0(\omega,\boldsymbol{\kappa}) &= \int \phi_0(\omega,\boldsymbol{x})\exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{x})d\boldsymbol{x} \end{aligned}$$

- Consider

$$2ik\partial_z\phi + \Delta_\perp\phi = 0, \qquad \phi(\omega, z=z_0, \boldsymbol{x}) = \exp\left(-\frac{|\boldsymbol{x}|^2}{r_0^2}\right)$$

We get

$$\phi(\omega,z,\boldsymbol{x}) = \frac{1}{\sqrt{1+\frac{2iz}{kr_0^2}}}\exp\left(-\frac{|\boldsymbol{x}|^2}{r_0^2\left(1+\frac{2iz}{kr_0^2}\right)}\right)$$

$\hookrightarrow$ Gaussian beam with radius $R(z) = \sqrt{1+\dfrac{4z^2}{k^2r_0^4}}$.

Transmitted wave - random case

$$2ik\partial_z\phi + \Delta_\perp\phi + \frac{k^2}{\varepsilon}\mu\Big(\frac{z}{\varepsilon^2}, \frac{\boldsymbol{x}}{\varepsilon^2}\Big)\phi = 0$$

In the regime $\varepsilon \ll 1$, the mean transmitted wave satisfies:

$$\begin{cases} \partial_z \mathbb{E}[\phi] = \dfrac{i}{2k}\Delta_\perp \mathbb{E}[\phi] - \dfrac{k^2\gamma_0}{8}\mathbb{E}[\phi], \\ \phi(z=0,\boldsymbol{x}) = \phi_0(\boldsymbol{x}), \end{cases}$$

with

$$\gamma_0 = 2\int_0^\infty \mathbb{E}[\mu(0,\mathbf{0})\mu(z,\mathbf{0})]dz$$

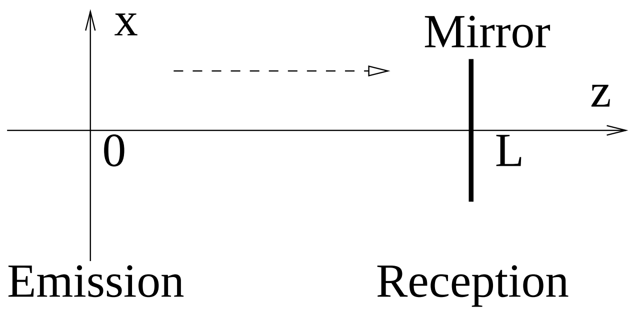
We find:

$$\mathbb{E}[\phi(\omega,z,\boldsymbol{x})] = \phi(\omega,z,\boldsymbol{x})\mid_{\text{homo}} \exp\Big(-\frac{\gamma_0\omega^2}{8c_0^2}z\Big)$$

$\hookrightarrow$ effective (frequency-dependent) damping.

What about fluctuations ?
we would like to study $\mathbb{E}[|\phi(\omega,z,\boldsymbol{x})|^2]$. But we need to look at $\mathbb{E}[\phi(\omega,z,\boldsymbol{x})\overline{\phi(\omega,z,\boldsymbol{y})}]$ to get a closed system.

Time reversal setup



1) Forward emission.

We emit at $z = 0$: $u_0(t, \boldsymbol{x}) = \dfrac{1}{2\pi} \displaystyle\int e^{-i\omega t} \hat{u}_0(\omega, \boldsymbol{x}) d\omega$.

We receive and record at $z = L$, $\boldsymbol{y} \in$ mirror:

$$u_F(t, L, \boldsymbol{y}) = \frac{1}{2\pi} \int e^{i\omega(\frac{L}{c_0} - t)} \phi_F(\omega, L, \boldsymbol{y}) d\omega$$

where

$$\phi_F(\omega, L, \boldsymbol{y}) = \int \hat{G}\big(\omega, (L, \boldsymbol{y}), (0, \boldsymbol{x})\big) \hat{u}_0(\omega, \boldsymbol{x}) dx$$

We shift the time origin:

$$\tilde{u}_F(t, L, \boldsymbol{y}) := u_F\Big(t - \frac{L}{c_0}, L, \boldsymbol{y}\Big) = \frac{1}{2\pi} \int e^{-i\omega t} \phi_F(\omega, L, \boldsymbol{y}) d\omega$$

2) Time reversal.

- The signal is recorded at the mirror (with limited spatial support).
- The recorded signal is time-reversed:

$$\begin{aligned} u_B(t, L, \boldsymbol{y}) &= \tilde{u}_F(-t, L, \boldsymbol{y}) \times \chi_M(\boldsymbol{y}) \\ &= \frac{1}{2\pi} \int e^{i\omega t} \phi_F(\omega, L, \boldsymbol{y}) d\omega \times \chi_M(\boldsymbol{y}) \end{aligned}$$

where χ_M is the (spatial) truncation function of the mirror.
If the mirror is square with side length D, $\chi_M(\boldsymbol{y}) = \mathbf{1}_{[-D/2,D/2]\times[-D/2,D/2]}(\boldsymbol{y})$.

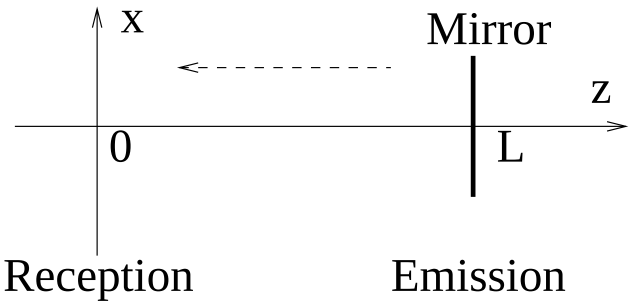
Taking the complex conjugate (u_B is real-valued):

$$u_B(t, L, \boldsymbol{y}) = \frac{1}{2\pi} \int e^{-i\omega t} \phi_B(\omega, L, \boldsymbol{y}) d\omega$$

with $\phi_B(\omega, L, \boldsymbol{y}) = \overline{\phi_F(\omega, L, \boldsymbol{y})} \chi_M(\boldsymbol{y})$

$$= \int \overline{\hat{G}\big(\omega, (L, \boldsymbol{y}), (0, \boldsymbol{x})\big) \hat{u}_0(\omega, \boldsymbol{x})} d\boldsymbol{x} \chi_M(\boldsymbol{y})$$

$\hookrightarrow$ The signal $u_B(t, L, \boldsymbol{y})$ is emitted from the mirror.



3) Reception of the backward signal.

We emit at $z = L$: $u_B(t, L, \boldsymbol{x}) = \dfrac{1}{2\pi} \displaystyle\int e^{-i\omega t} \phi_B(\omega, L, \boldsymbol{x}) d\omega.$

We receive at $z = 0$: $u_B(t, 0, \boldsymbol{x}) = \dfrac{1}{2\pi} \displaystyle\int e^{i\omega(\frac{L}{c_0} - t)} \phi_B(\omega, 0, \boldsymbol{x}) d\omega.$

We shift the time origin: $\tilde{u}_B(t, 0, \boldsymbol{x}) := u_B\Big(t - \dfrac{L}{c_0}, 0, \boldsymbol{x}\Big) = \dfrac{1}{2\pi} \displaystyle\int e^{-i\omega t} \phi_B(\omega, 0, \boldsymbol{x}) d\omega$

where

$$
\begin{aligned}
\phi_B(\omega, 0, \boldsymbol{x}) &= \int \hat{G}\big(\omega, (0, \boldsymbol{x}), (L, \boldsymbol{y})\big) \phi_B(\omega, L, \boldsymbol{y}) d\boldsymbol{y} \\
&= \int \int \hat{G}\big(\omega, (0, \boldsymbol{x}), (L, \boldsymbol{y})\big) \overline{\hat{G}\big(\omega, (L, \boldsymbol{y}), (0, \boldsymbol{\eta})\big) \hat{u}_0(\omega, \boldsymbol{\eta})} \chi_M(\boldsymbol{y}) d\boldsymbol{\eta} d\boldsymbol{y}
\end{aligned}
$$

Reciprocity relation: $\hat{G}\big(\omega, (0, \boldsymbol{x}), (L, \boldsymbol{y})\big) = \hat{G}\big(\omega, (L, \boldsymbol{y}), (0, \boldsymbol{x})\big).$

Finally, we receive the signal

$$\tilde{u}_B(t,0,\boldsymbol{x}) = \frac{1}{2\pi}\int e^{-i\omega t}\phi_B(\omega,0,\boldsymbol{x})d\omega$$

with

$$\phi_B(\omega,0,\boldsymbol{\xi}) = \int\int \Gamma(\omega,L,\boldsymbol{y},\boldsymbol{y},\boldsymbol{\xi},\boldsymbol{\eta})\overline{\hat{u}_0(\omega,\boldsymbol{\eta})}\chi_M(\boldsymbol{y})d\boldsymbol{\eta}d\boldsymbol{y}$$

where

$$\Gamma(\omega,L,\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi},\boldsymbol{\eta}) := \hat{G}\big(\omega,(L,\boldsymbol{x}),(0,\boldsymbol{\xi})\big)\overline{\hat{G}\big(\omega,(L,\boldsymbol{y}),(0,\boldsymbol{\eta})\big)}$$

- In a homogeneous medium, Γ is explicitly known $\Rightarrow$ the double integral can be computed.
- In random media: the statistical distribution of Γ is important.

Wigner of Γ

$\Gamma(\omega, L, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}, \boldsymbol{\eta}) := \hat{G}\big(\omega, (L, \boldsymbol{x}), (0, \boldsymbol{\xi})\big)\overline{\hat{G}\big(\omega, (L, \boldsymbol{y}), (0, \boldsymbol{\eta})\big)}$ is solution of

$$2ik\frac{\partial \Gamma}{\partial L} + (\Delta_{\boldsymbol{x}} - \Delta_{\boldsymbol{y}})\Gamma + k^2(\mu(L, \boldsymbol{x}) - \mu(L, \boldsymbol{y}))\Gamma = 0$$

starting from $\Gamma(\omega, L = 0, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \delta(\boldsymbol{x} - \boldsymbol{\xi})\delta(\boldsymbol{y} - \boldsymbol{\eta})$.

Change of variables: $\boldsymbol{X} = (\boldsymbol{x} + \boldsymbol{y})/2$ and $\boldsymbol{Y} = \boldsymbol{x} - \boldsymbol{y}$.

$$2ik\frac{\partial \Gamma}{\partial L} + 2\nabla_{\boldsymbol{X}} \cdot \nabla_{\boldsymbol{Y}}\Gamma + k^2\Big(\mu\big(L, \boldsymbol{X} + \frac{\boldsymbol{Y}}{2}\big) - \mu\big(L, \boldsymbol{X} - \frac{\boldsymbol{Y}}{2}\big)\Big)\Gamma = 0$$

We introduce:

$$W(\omega, L, \boldsymbol{X}, \boldsymbol{P}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \int e^{-i\boldsymbol{P}\cdot\boldsymbol{Y}}\Gamma(\omega, L, \boldsymbol{X} + \frac{\boldsymbol{Y}}{2}, \boldsymbol{X} - \frac{\boldsymbol{Y}}{2}, \boldsymbol{\xi}, \boldsymbol{\eta})d\boldsymbol{Y}$$

Γ is obtained by inverse transform:

$$\Gamma(\omega, L, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{(2\pi)^{d-1}}\int W(\omega, L, \frac{\boldsymbol{x} + \boldsymbol{y}}{2}, \boldsymbol{P}, \boldsymbol{\xi}, \boldsymbol{\eta})e^{i\boldsymbol{P}\cdot(\boldsymbol{x}-\boldsymbol{y})}d\boldsymbol{P}$$

$$\Gamma(\omega, L, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{(2\pi)^{d-1}}\int W(\omega, L, \boldsymbol{y}, \boldsymbol{P}, \boldsymbol{\xi}, \boldsymbol{\eta})d\boldsymbol{P}$$

W is solution of

$$k\frac{\partial W}{\partial L} + \boldsymbol{P} \cdot \nabla_{\boldsymbol{X}} W = \mathcal{L} W$$

$$\mathcal{L}W = -\frac{ik^2}{2(2\pi)^{d-1}} \int e^{i\boldsymbol{Q}\cdot\boldsymbol{X}} \hat{\mu}(L,\boldsymbol{Q}) \Big[W\big(L,\boldsymbol{X},\boldsymbol{P}+\frac{\boldsymbol{Q}}{2}\big) - W\big(L,\boldsymbol{X},\boldsymbol{P}-\frac{\boldsymbol{Q}}{2}\big) \Big] d\boldsymbol{Q}$$

starting from

$$W(L=0,\boldsymbol{X},\boldsymbol{P}) = e^{-i\boldsymbol{P}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})} \delta\left(\boldsymbol{X} - \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)$$

where $\hat{\mu}$ is the Fourier transform of μ w.r.t. $\boldsymbol{X}$:

$$\hat{\mu}(L,\boldsymbol{Q}) = \int e^{-i\boldsymbol{Q}\cdot\boldsymbol{X}} \mu(L,\boldsymbol{X}) d\boldsymbol{X}$$

Homogeneous medium $\mu \equiv 0$

Solution: $W(L, \boldsymbol{X}, \boldsymbol{P}) = W\big(z=0, \boldsymbol{X} - \frac{L}{k}\boldsymbol{P}, \boldsymbol{P}\big) = e^{-i\boldsymbol{P}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})}\delta\left(\boldsymbol{X} - \frac{L\boldsymbol{P}}{k} - \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)$.

Thus $\Gamma(L, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{k^{d-1}}{(2\pi L)^{d-1}} e^{-i\frac{k}{L}(\boldsymbol{y}-\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})}$.

If $u_0(t, \boldsymbol{x}) = f(t)\delta_0(\boldsymbol{x})$, then $\phi_B(\omega, 0, \boldsymbol{x}) = \frac{\omega^2}{(2\pi c_0 L)^2}\overline{\hat{f}(\omega)} e^{i\frac{\omega|\boldsymbol{x}|^2}{2Lc_0}} \hat{\chi}_M\left(\frac{\omega \boldsymbol{x}}{c_0 L}\right)$.

In case of a square mirror:

$$\phi_B(\omega, 0, \boldsymbol{x}) = \frac{(\omega D)^2}{(2\pi c_0 L)^2}\overline{\hat{f}(\omega)} e^{i\frac{\omega|\boldsymbol{x}|^2}{2Lc_0}} \operatorname{sinc}\left(\frac{\pi x_1}{r_c(\omega)}\right) \operatorname{sinc}\left(\frac{\pi x_2}{r_c(\omega)}\right)$$

where $r_c(\omega) = \frac{2\pi L c_0}{\omega D}$ and $\operatorname{sinc}(s) = \sin(s)/s$.

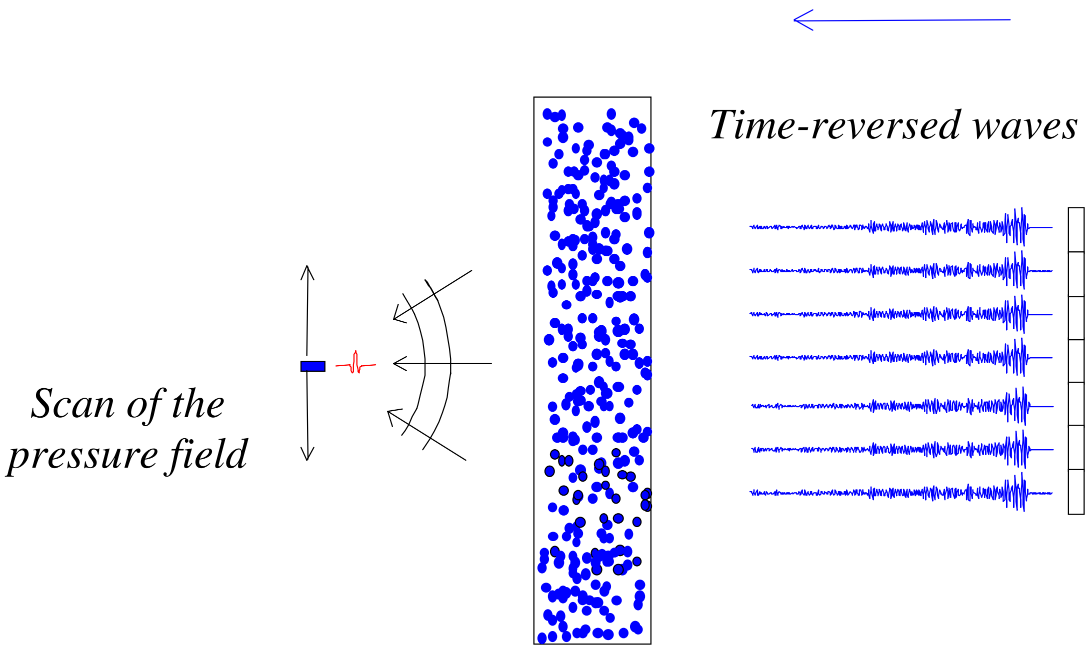
In the time domain:

if $u_0(t, \boldsymbol{x}) = \cos(\omega_0 t) v(t) \delta_0(\boldsymbol{x})$ with bandwidth$(v) \ll \omega_0$, then:

$$u_B(t, 0, \boldsymbol{x}) = \frac{(\omega_0 D)^2}{(2\pi c_0 L)^2}\cos(\omega_0 t) v\left(-t - \frac{|\boldsymbol{x}|^2}{2Lc_0}\right) \operatorname{sinc}\left(\frac{\pi x_1}{r_c}\right) \operatorname{sinc}\left(\frac{\pi x_2}{r_c}\right)$$

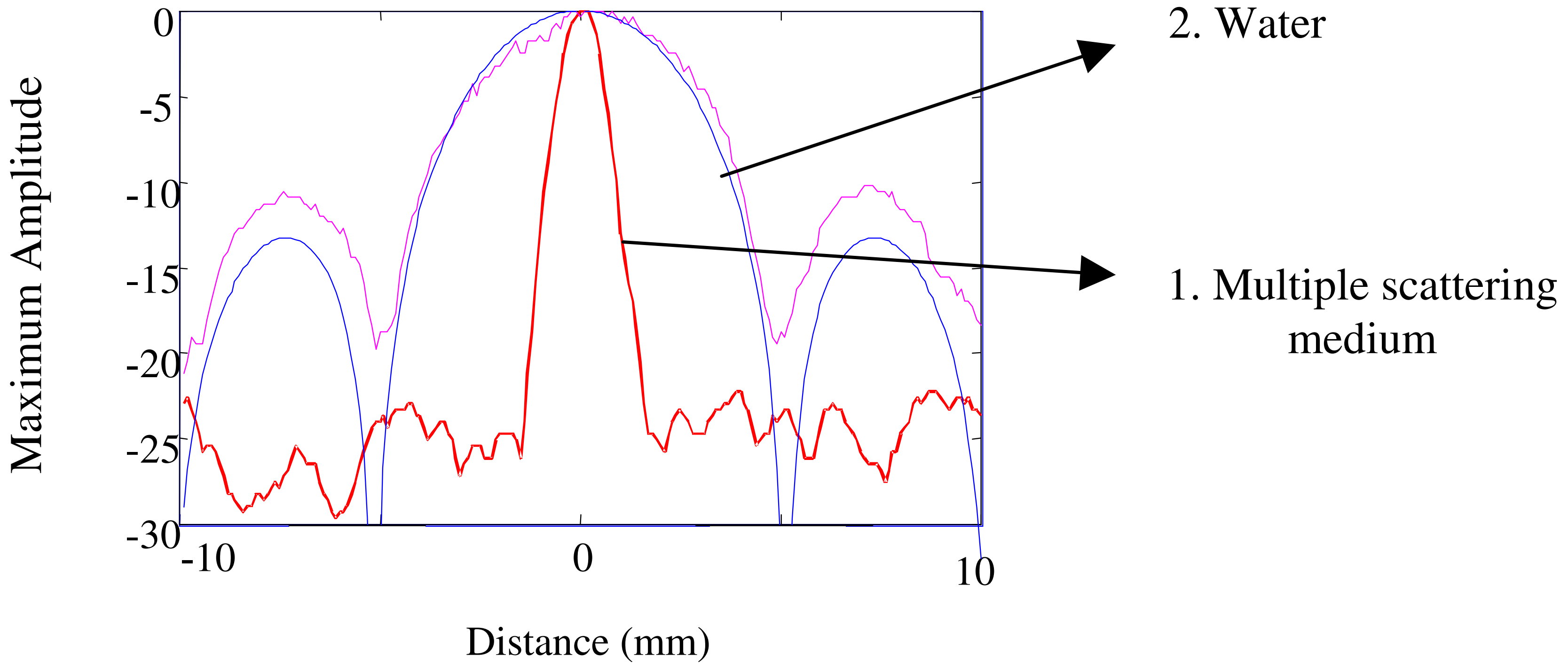
where $r_c = \frac{2\pi L c_0}{\omega_0 D} = \frac{\lambda_0 L}{D}$.

(*a*)



Time-reversal experiment

(*b*)



Scaling and limit theorem

Weakly perturbed medium $\mu \to \varepsilon\mu$.

Re-scale $L \to L/\varepsilon^2$ and $\boldsymbol{x} \to \boldsymbol{x}/\varepsilon^2$

W^ε is solution of

$$k\frac{\partial W^\varepsilon}{\partial L} + \boldsymbol{P}\cdot\nabla_{\boldsymbol{X}} W^\varepsilon = \mathcal{L}^\varepsilon W^\varepsilon$$

$$\mathcal{L}^\varepsilon W^\varepsilon = -\frac{ik^2}{2(2\pi)^d\varepsilon}\int e^{i\boldsymbol{Q}\cdot\frac{\boldsymbol{X}}{\varepsilon^2}}\hat{\mu}\big(\frac{L}{\varepsilon^2},\boldsymbol{Q}\big)\Big[W^\varepsilon\big(\boldsymbol{X},\boldsymbol{P}+\frac{\boldsymbol{Q}}{2}\big) - W^\varepsilon\big(\boldsymbol{X},\boldsymbol{P}-\frac{\boldsymbol{Q}}{2}\big)\Big]\,d\boldsymbol{Q}$$

Result: $\mathbb{E}\left[W^{\varepsilon}\right] \to \bar{W}(L, \boldsymbol{X}, \boldsymbol{P})$ solution of

$$k\frac{\partial \bar{W}}{\partial L} + \boldsymbol{P}\cdot\nabla_{\boldsymbol{X}}\bar{W} = \bar{\mathcal{L}}\bar{W}$$

$$\bar{\mathcal{L}}\bar{W} = \frac{k^3}{4(2\pi)^3}\int \hat{R}\Big(\frac{|\boldsymbol{P}|^2 - |\boldsymbol{Q}|^2}{2k}, \boldsymbol{P}-\boldsymbol{Q}\Big)\left(\bar{W}(\boldsymbol{Q}) - \bar{W}(\boldsymbol{P})\right) d\boldsymbol{Q}$$

where $\quad \hat{R}(\kappa, \boldsymbol{Q}) = \int R(z, \boldsymbol{X}) e^{-i\boldsymbol{Q}\cdot\boldsymbol{X} + i\kappa z} d\boldsymbol{X}\, dz,$

$$R(z, \boldsymbol{X}) = \mathbb{E}\left[\mu(0, \mathbf{0})\mu(z, \boldsymbol{X})\right].$$

Note: $\hat{R}$ is nonnegative-valued and even.

The proof is “easy” because of the explicit mixing w.r.t. L.

Standard results on radiative transport

$$i\frac{\partial\phi}{\partial t}+\frac{1}{2}\Delta_{\boldsymbol{x}}\phi-V(\boldsymbol{x})\phi=0,\quad \boldsymbol{x}\in\mathbb{R}^d,\ t>0$$

Usually $d=3$. Start from $\phi(t=0,\boldsymbol{x})=\phi_0(\boldsymbol{x})$ (smooth).

Wigner: $W(\boldsymbol{x},\boldsymbol{p})=\int e^{-i\boldsymbol{p}\cdot\boldsymbol{y}}\phi(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y})\overline{\phi(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y})}dy$

Property: $\frac{1}{(2\pi)^d}\int W(\boldsymbol{x},\boldsymbol{p})d\boldsymbol{p}=|\phi(\boldsymbol{x})|^2$

W is solution of:

$$\frac{\partial W}{\partial t}+\boldsymbol{p}\cdot\nabla_{\boldsymbol{x}}W+\mathcal{L}_{\boldsymbol{x}}W=0$$

$$\mathcal{L}_{\boldsymbol{x}}W=-\frac{i}{(2\pi)^d}\int e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\hat{V}(\boldsymbol{q})\Big(W\big(\boldsymbol{x},\boldsymbol{p}+\frac{\boldsymbol{q}}{2}\big)-W\big(\boldsymbol{x},\boldsymbol{p}-\frac{\boldsymbol{q}}{2}\big)\Big)d\boldsymbol{q}$$

Re-scale: $V \to \varepsilon V$, $t \to t/\varepsilon^2$ and $\boldsymbol{x} \to \boldsymbol{x}/\varepsilon^2$

$$i\varepsilon^2 \frac{\partial \phi^\varepsilon}{\partial t} + \frac{\varepsilon^4}{2} \Delta_{\boldsymbol{x}} \phi^\varepsilon - \varepsilon V(\frac{\boldsymbol{x}}{\varepsilon^2})\phi^\varepsilon = 0, \quad \phi^\varepsilon(t, \boldsymbol{x}) = \phi\big(\frac{t}{\varepsilon^2}, \frac{\boldsymbol{x}}{\varepsilon^2}\big)$$

Wigner: $W^\varepsilon(\boldsymbol{x}, \boldsymbol{p}) = \displaystyle\int e^{-i\boldsymbol{p}\cdot\boldsymbol{y}} \phi^\varepsilon\big(\boldsymbol{x} + \frac{\varepsilon^2}{2}\boldsymbol{y}\big) \overline{\phi^\varepsilon\big(\boldsymbol{x} - \frac{\varepsilon^2}{2}\boldsymbol{y}\big)} dy$

W^ε is solution of:

$$\frac{\partial W^\varepsilon}{\partial t} + \boldsymbol{p} \cdot \nabla_{\boldsymbol{x}} W^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}_{\frac{\boldsymbol{x}}{\varepsilon^2}} W^\varepsilon = 0$$

Result: $\mathbb{E}\left[W^\varepsilon\right] \to \bar{W}$ solution of

$$\frac{\partial \bar{W}}{\partial t} + \boldsymbol{p} \cdot \bar{W} = \bar{\mathcal{L}} \bar{W}$$

$$\bar{\mathcal{L}} \bar{W} = \frac{4\pi}{(2\pi)^d} \int \hat{R}(\boldsymbol{q} - \boldsymbol{p}) \delta(|\boldsymbol{q}|^2 - |\boldsymbol{p}|^2) \left(\bar{W}(\boldsymbol{x}, \boldsymbol{q}) - \bar{W}(\boldsymbol{x}, \boldsymbol{p})\right) d\boldsymbol{q}$$

where $\hat{R}(\boldsymbol{q}) = \displaystyle\int \mathbb{E}\left[V(\boldsymbol{x}) V(\boldsymbol{x} + \boldsymbol{y})\right] e^{-i\boldsymbol{q}\cdot\boldsymbol{y}} d\boldsymbol{y}$

The proof is "difficult" because V does not depend explicitly on t (while μ depended explicitly on L).

Back to W = Wigner of Γ

W^ε is solution of

$$k\frac{\partial W^\varepsilon}{\partial L} + \boldsymbol{P}\cdot\nabla_{\boldsymbol{X}} W^\varepsilon = \mathcal{L}^\varepsilon W^\varepsilon$$

with

$$\mathcal{L}^\varepsilon W^\varepsilon = -\frac{ik^2}{2(2\pi)^d\varepsilon}\int e^{i\boldsymbol{Q}\cdot\frac{\boldsymbol{X}}{\varepsilon^2}}\hat{\mu}\Big(\frac{L}{\varepsilon^2},\boldsymbol{Q}\Big)\Big[W^\varepsilon\Big(\boldsymbol{X},\boldsymbol{P}+\frac{\boldsymbol{Q}}{2}\Big) - W^\varepsilon\Big(\boldsymbol{X},\boldsymbol{P}-\frac{\boldsymbol{Q}}{2}\Big)\Big]\,d\boldsymbol{Q}$$

Result: $\mathbb{E}\left[W^\varepsilon\right] \to \bar{W}(L,\boldsymbol{X},\boldsymbol{P})$ solution of

$$k\frac{\partial \bar{W}}{\partial L} + \boldsymbol{P}\cdot\nabla_{\boldsymbol{X}} \bar{W} = \bar{\mathcal{L}}\bar{W}$$

$$\bar{\mathcal{L}}\bar{W} = \frac{k^3}{4(2\pi)^3}\int \hat{R}\Big(\frac{|\boldsymbol{P}|^2-|\boldsymbol{Q}|^2}{2k},\boldsymbol{P}-\boldsymbol{Q}\Big)\left(\bar{W}(\boldsymbol{Q})-\bar{W}(\boldsymbol{P})\right)d\boldsymbol{Q}$$

where $\quad \hat{R}(\kappa,\boldsymbol{Q}) = \int R(z,\boldsymbol{X})e^{-i\boldsymbol{Q}\cdot\boldsymbol{X}+i\kappa z}d\boldsymbol{X}\,dz,$

$\quad R(z,\boldsymbol{X}) = \mathbb{E}\left[\mu(0,\mathbf{0})\mu(z,\boldsymbol{X})\right].$

Simplification of the effective transport equation

$$k\frac{\partial \bar{W}}{\partial L} + \boldsymbol{P}\cdot\nabla_{\boldsymbol{X}}\bar{W} = \bar{\mathcal{L}}\bar{W}$$

$$\bar{\mathcal{L}}\bar{W} = \frac{k^3}{4(2\pi)^3}\int \hat{R}\Big(\frac{|\boldsymbol{P}|^2-|\boldsymbol{Q}|^2}{2k}, \boldsymbol{P}-\boldsymbol{Q}\Big)\left(\bar{W}(\boldsymbol{Q})-\bar{W}(\boldsymbol{P})\right)d\boldsymbol{Q}$$

where $\hat{R}(\kappa,\boldsymbol{Q}) = \int R(z,\boldsymbol{X})e^{i\boldsymbol{Q}\cdot\boldsymbol{X}+i\kappa z}d\boldsymbol{X}dz$, $R(z,\boldsymbol{X}) = \mathbb{E}\left[\mu(0,\mathbf{0})\mu(z,\boldsymbol{X})\right]$.

Approximation: $R(z,\boldsymbol{X}) = m(z/z_c)r(\boldsymbol{X}/l_c)$ with $l_c \gg z_c$. Then
$\hat{R}(\kappa,\boldsymbol{Q}) = z_c\hat{m}(\kappa z_c)l_c^2\hat{r}(\boldsymbol{Q}l_c) \simeq \sigma^2 l_c^2\hat{r}(Ql_c)$ where $\sigma^2 = z_c\hat{m}(0)$.

$$\begin{aligned}
\bar{\mathcal{L}}\bar{W} &= \frac{k^3}{4(2\pi)^3}\int \sigma^2 l_c^2\hat{r}(l_c(\boldsymbol{P}-\boldsymbol{Q}))\left(\bar{W}(\boldsymbol{Q})-\bar{W}(\boldsymbol{P})\right)d\boldsymbol{Q}\\
&= \frac{k^3}{4(2\pi)^3}\sigma^2\int \hat{r}(\boldsymbol{Q}')\left(\bar{W}(\boldsymbol{P}+\boldsymbol{Q}'/l_c)-\bar{W}(\boldsymbol{P})\right)d\boldsymbol{Q}'\\
&= \frac{k^3}{4(2\pi)^3}\sigma^2\int \hat{r}(\boldsymbol{Q}')\left(\nabla\bar{W}(\boldsymbol{P})\cdot\boldsymbol{Q}'/l_c + \frac{1}{2}\boldsymbol{Q}'\cdot\nabla\nabla\bar{W}(\boldsymbol{P})\boldsymbol{Q}'/l_c^2 + \ldots\right)d\boldsymbol{Q}'
\end{aligned}$$

The first term is zero ($\hat{r}$ is even). The second term is:

$$\bar{\mathcal{L}}\bar{W} = \frac{k^3}{16}\gamma_2\Delta_{\boldsymbol{P}}\bar{W}$$

with $\gamma_2 = \frac{\sigma^2}{(2\pi)^3 l_c^2}\int |\boldsymbol{Q}'|^2\hat{r}(\boldsymbol{Q}')d\boldsymbol{Q}' = -\int \Delta_{\boldsymbol{X}}R(z,\mathbf{0})dz \geq 0$

Resolution of the effective transport equation

$$\frac{\partial \bar{W}}{\partial L} = -\frac{\boldsymbol{P}}{k} \cdot \nabla_{\boldsymbol{X}} \bar{W} + \frac{k^2 \gamma_2}{16} \Delta_{\boldsymbol{P}} \bar{W}$$

starting from $\bar{W}(L=0, \boldsymbol{X}, \boldsymbol{P}) = e^{-i\boldsymbol{P}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})} \delta\left(\boldsymbol{X} - \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)$

In order to solve this partial differential equation:

Method 1: Probabilistic interpretation

$$\begin{cases} d\boldsymbol{P}(L) = \dfrac{\sqrt{k^2\gamma_2}}{\sqrt{8}} d\boldsymbol{B}_L, & \boldsymbol{P}(0) = \boldsymbol{P} \\ d\boldsymbol{X}(L) = -\dfrac{\boldsymbol{P}(L)}{k} dL, & \boldsymbol{X}(0) = \boldsymbol{X} \end{cases}$$

and then $\bar{W}(L, \boldsymbol{X}, \boldsymbol{P}) = \mathbb{E}\left[e^{-i\boldsymbol{P}(L)\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})} \delta\left(\boldsymbol{X}(L) - \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2} \right) \right]$

Method 2: Fourier calculations: take a Fourier transform in $\boldsymbol{P}$ and $\boldsymbol{X}$ and solve.

Thus $\bar{\Gamma}(L, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \underbrace{\frac{k^2}{(2\pi L)^2} e^{-i\frac{k}{L}\left(\boldsymbol{y}-\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})}}_{\text{homogeneous case}} e^{-\frac{k^2\gamma_2 L}{48}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2}.$

Transmitted wave

Initial conditions: $u_0(t,\boldsymbol{x}) = \phi_0(\boldsymbol{x}) f(t)$, $\hat{u}_0(\omega,\boldsymbol{x}) = \phi_0(\boldsymbol{x})\hat{f}(\omega)$.

We have already seen that the mean transmitted wave is

$$\mathbb{E}[\phi_F(\omega,L,\boldsymbol{x})] = \phi_F(\omega,L,\boldsymbol{x}) \mid_{\text{homo}} \exp\Big(-\frac{\gamma_0\omega^2}{8c_0^2}L\Big)$$

with

$$\gamma_0 = \int_{-\infty}^{\infty} R(z,\mathbf{0})dz$$

Consider now the mean transmitted intensity:

$$\begin{aligned}
\mathbb{E}[|\phi_F(\omega,L,\boldsymbol{x})|^2] &= \iint \mathbb{E}\big[\hat{G}\big(\omega,(L,\boldsymbol{x}),(\boldsymbol{\xi},0)\big)\overline{\hat{G}\big(\omega,(L,\boldsymbol{x}),(0,\boldsymbol{\eta})\big)}\big]\phi_0(\boldsymbol{\xi})\phi_0(\boldsymbol{\eta})d\boldsymbol{\eta}d\boldsymbol{\xi} \\
&= \iint \bar{\Gamma}(\omega,L,\boldsymbol{x},\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{\eta})\phi_0(\boldsymbol{\xi})\overline{\phi_0(\boldsymbol{\eta})}d\boldsymbol{\eta}d\boldsymbol{\xi} \\
&= \iint \Gamma(\omega,L,\boldsymbol{x},\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{\eta}) \mid_{\text{homo}} e^{-\frac{\omega^2\gamma_2 L}{48c_0^2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2}\phi_0(\boldsymbol{\xi})\overline{\phi_0(\boldsymbol{\eta})}d\boldsymbol{\eta}d\boldsymbol{\xi}
\end{aligned}$$

If the radius of ϕ_0 is smaller than $r_a = \frac{\sqrt{24}c_0}{\omega\sqrt{\gamma_2 L}}$, then

$$\mathbb{E}[|\phi_F(\omega,L,\boldsymbol{x})|^2] = \Big|\phi_F(\omega,L,\boldsymbol{x}) \mid_{\text{homo}}\Big|^2$$

If $\phi_0(\boldsymbol{x}) = \exp(-|\boldsymbol{x}|^2/r_0^2)$, then

$$\mathbb{E}[\phi_F(\omega, L, \boldsymbol{x})] = \frac{1}{\left(1 + \frac{4L^2}{k^2 r_0^4}\right)^{\frac{d-1}{4}}} \exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2\left(1+\frac{4L^2}{k^2 r_0^4}\right)}\Big) \exp\Big(-\frac{\gamma_0 \omega^2}{8 c_0^2} L\Big)$$

with $\gamma_0 = \int_{-\infty}^{\infty} R(z, \mathbf{0}) dz$.

$$\mathbb{E}[|\phi_F(\omega, L, \boldsymbol{x})|^2] = \frac{1}{\left(1 + \frac{4L^2}{k^2 r_0^4}(1 + \frac{r_0^2}{r_a^2})\right)^{\frac{d-1}{2}}} \exp\Big(-\frac{2|\boldsymbol{x}|^2}{r_0^2\left(1+\frac{4L^2}{k^2 r_0^4}(1+\frac{r_0^2}{r_a^2})\right)}\Big)$$

with $r_a = \frac{\sqrt{24} c_0}{\omega \sqrt{\gamma_2 L}}$ and $\gamma_2 = -\int_{-\infty}^{\infty} \Delta_{\boldsymbol{x}} R(z, \mathbf{0}) dz$.

Note that the radius of the transmitted intensity grows like $L^{3/2}$.

Refocused wave

With: $u_0(t,\boldsymbol{x}) = \delta(\boldsymbol{x}) f(t)$, $\hat{u}_0(\omega,\boldsymbol{x}) = \delta(\boldsymbol{x})\hat{f}(\omega)$.

$$\begin{aligned}
\mathbb{E}\left[\phi_B(\omega,0,\boldsymbol{x})\right] &= \iint \bar{\Gamma}(\omega,L,\boldsymbol{y},\boldsymbol{y},\boldsymbol{x},\boldsymbol{\eta})\overline{\delta(\boldsymbol{\eta})}\chi_M(\boldsymbol{y})d\boldsymbol{\eta}d\boldsymbol{y} \\
&= \int \Gamma(\omega,L,\boldsymbol{y},\boldsymbol{y},\boldsymbol{x},\boldsymbol{0})\mid_{\text{homo}} e^{-\frac{\omega^2\gamma_2 L}{48c_0^2}|\boldsymbol{x}|^2}\chi_M(\boldsymbol{y})d\boldsymbol{y} \\
&= \phi_B(\omega,0,\boldsymbol{x})\mid_{\text{homo}} e^{-\frac{\omega^2\gamma_2 L}{48c_0^2}|\boldsymbol{x}|^2}
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\phi_B(\omega,0,\boldsymbol{x})\right] = \underbrace{\frac{\omega^2}{(2\pi c_0 L)^2}\overline{\hat{f}(\omega)}e^{i\frac{\omega|\boldsymbol{x}|^2}{2Lc_0}}\operatorname{sinc}\left(\frac{x_1}{r_c}\right)\operatorname{sinc}\left(\frac{x_2}{r_c}\right)}_{\text{homogeneous case}}\exp\left(-\frac{|\boldsymbol{x}|^2}{2r_a^2}\right)$$

where $r_c = \dfrac{2\pi L c_0}{\omega D}$ and $r_a = \dfrac{\sqrt{24}c_0}{\omega\sqrt{\gamma_2 L}}$.

In the time domain:

$u_0(t, \boldsymbol{x}) = \cos(\omega_0 t) v(t) \delta(\boldsymbol{x})$ with bandwidth$(v) \ll \omega_0$:

$$\mathbb{E}\left[u_B(t,0,\boldsymbol{x})\right] = \cos(\omega_0 t) v\left(-t - \frac{|\boldsymbol{x}|^2}{2Lc_0}\right) \operatorname{sinc}\left(\frac{x_1}{r_c}\right) \operatorname{sinc}\left(\frac{x_2}{r_c}\right) \boxed{\exp\left(-\frac{|\boldsymbol{x}|^2}{2r_a^2}\right)}$$

where $r_c = \dfrac{2\pi L c_0}{\omega_0 D} = \dfrac{\lambda_0 L}{D}$ and $r_a = \dfrac{\sqrt{24} c_0}{\omega_0 \sqrt{\gamma_2 L}} = \dfrac{\lambda_0 \sqrt{6}}{\pi \sqrt{\gamma_2 L}}$.

If $r_c < r_a$: Same result as in homogeneous medium

If $r_c > r_a$: $\mathbb{E}\left[u_B\right]$ spot thiner than in homogeneous medium !

The radius of the focal spot is

$$r_{\text{eff}} = \frac{\lambda_0 L}{D_{\text{eff}}}$$

with the effective aperture: $D_{\text{eff}}^2 = D^2 + \frac{\pi^2}{2} \gamma_2 L^3$.

But: the result holds true in average (averaging over all possible realizations of the medium) !

Frequency correlation of W

We have shown that ($\varepsilon \to 0$):

$$\mathbb{E}\left[\phi_B(\omega, 0, \boldsymbol{x})\right] = \int\int\int \bar{W}(\omega, L, \boldsymbol{y}, \boldsymbol{P}, \boldsymbol{x}, \boldsymbol{\eta}) \overline{\hat{u}_0(\omega, \boldsymbol{\eta})} \chi_M(\boldsymbol{y}) d\boldsymbol{y} d\boldsymbol{\eta} d\boldsymbol{P}$$

where $\bar{W} = \lim_{\varepsilon \to 0} \mathbb{E}\left[W^\varepsilon\right]$.

The frequency autocorrelation function of W^ε:

$$\mathbb{E}\left[W^\varepsilon(\omega + \varepsilon h/2, L, \ldots) W^\varepsilon(\omega - \varepsilon h/2, L, \ldots)\right] \overset{\varepsilon \to 0}{\longrightarrow} \gamma(\omega, h, L, \ldots)$$

with $\gamma(\omega, h, L, \ldots) \overset{h \to \infty}{\longrightarrow} 0$.

Thus

$$\mathbb{E}\left[W^\varepsilon(\omega_1, L, \ldots) W^\varepsilon(\omega_2, L, \ldots)\right] \simeq \bar{W}(\omega_1, L, \ldots) \bar{W}(\omega_2, L, \ldots)$$

as soon as $|\omega_1 - \omega_2| \gg \varepsilon$.

Similarly

$$\mathbb{E}\left[\phi_B(\omega_1, 0, \boldsymbol{x}) \phi_B(\omega_2, 0, \boldsymbol{x})\right] \simeq \mathbb{E}\left[\phi_B(\omega_1, 0, \boldsymbol{x})\right] \mathbb{E}\left[\phi_B(\omega_2, 0, \boldsymbol{x})\right]$$

as soon as $|\omega_1 - \omega_2| \gg \varepsilon$.

Self-averaging in time

In the time domain: $\mathbb{E}\left[u_B(t,\boldsymbol{x})\right] = \frac{1}{2\pi}\int e^{-i\omega t}\mathbb{E}\left[\phi_B(\omega,\boldsymbol{x})\right]d\omega,$

$$\mathbb{E}\left[u_B(t,\boldsymbol{x})^2\right] = \frac{1}{(2\pi)^2}\mathbb{E}\left[\left(\int e^{-i\omega t}\phi_B(\omega,\boldsymbol{x})d\omega\right)^2\right]$$

$$\begin{aligned}
\mathbb{E}\left[u_B(t,\boldsymbol{x})^2\right] &= \frac{1}{(2\pi)^2}\int\int e^{-i(\omega_1+\omega_2)t}\mathbb{E}\left[\phi_B(\omega_1,\boldsymbol{x})\phi_B(\omega_2,\boldsymbol{x})\right]d\omega_1 d\omega_2 \\
&\overset{\varepsilon\to 0}{\simeq} \frac{1}{(2\pi)^2}\int\int e^{-i(\omega_1+\omega_2)t}\mathbb{E}\left[\phi_B(\omega_1,\boldsymbol{x})\right]\mathbb{E}\left[\phi_B(\omega_2,\boldsymbol{x})\right]d\omega_1 d\omega_2 \\
&= \mathbb{E}\left[u_B(t,\boldsymbol{x})\right]^2
\end{aligned}$$

Thus $\mathrm{Var}(u_B(t,\boldsymbol{x})) := \mathbb{E}\Big[\big(u_B(t,\boldsymbol{x}) - \mathbb{E}\left[u_B(t,\boldsymbol{x})\right]\big)^2\Big] \overset{\varepsilon\to 0}{\longrightarrow} 0.$

This implies that, for any $\delta > 0$:

$$\mathbb{P}\Big(\left|u_B(t,\boldsymbol{x}) - \mathbb{E}\left[u_B(t,\boldsymbol{x})\right]\right| > \delta\Big) \leq \frac{\mathrm{Var}\left(u_B(t,\boldsymbol{x})\right)}{\delta^2} \overset{\varepsilon\to 0}{\longrightarrow} 0$$

The frequency decorrelation implies the self-averaging in time.

Conclusion

- Spatial time-reversal refocusing enhanced by randomness:

- in the homogeneous case, the diameter of the focal spot $\lambda L/D$ is determined by the aperture of the focusing cone (the waves propagate in a straightforward manner).
- in the random case, the focal spot is smaller because it is determined by an effective focusing cone induced by multi-pathing.

- Statistical stability of the refocused focal spot ensured by the frequency decorrelation of Γ or W.

The split-step Fourier method

Efficient numerical scheme to solve the Schrödinger equation:

$$2ik\partial_z\phi + \Delta_\perp\phi + k^2\mu(z,\boldsymbol{x})\phi = 0$$

with initial conditions $\phi(z=0,\boldsymbol{x}) = \phi_0(\boldsymbol{x})$.

Write:

$$\partial_z\phi = \mathcal{L}\phi + \mathcal{V}\phi$$

with

$$\mathcal{L}\phi = \frac{i}{2k}\Delta_\perp\phi, \qquad \mathcal{V}\phi = \frac{ik}{2}\mu\phi$$

- Remark 1: the equation

$$\partial_z \phi = \mathcal{L}\phi, \qquad \phi(z=0,\boldsymbol{x}) = \phi_0(\boldsymbol{x})$$

can be solved analytically because it can be reduced to an ordinary differential equation in the Fourier domain:

$$\partial_z \hat{\phi}(z,\boldsymbol{\kappa}) = -\frac{i|\boldsymbol{\kappa}|^2}{2k}\hat{\phi}(z,\boldsymbol{\kappa}), \qquad \hat{\phi}(z=0,\boldsymbol{\kappa}) = \hat{\phi}_0(\boldsymbol{\kappa})$$

$\hookrightarrow$

$$\begin{aligned}\phi(z,\boldsymbol{x}) = e^{\mathcal{L}z}\phi_0(\boldsymbol{x}) \quad &= \quad IFT\Big(e^{\frac{-i|\boldsymbol{\kappa}|^2 z}{2k}} DFT(\phi_0)(\boldsymbol{\kappa})\Big)(\boldsymbol{x}) \\ &= \quad \frac{1}{(2\pi)^{d-1}}\int e^{i\boldsymbol{\kappa}\cdot\boldsymbol{x}} e^{\frac{-i|\boldsymbol{\kappa}|^2 z}{2k}} \int e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{x}'}\phi_0(\boldsymbol{x}')d\boldsymbol{x}'d\boldsymbol{\kappa}\end{aligned}$$

- Remark 2: the equation

$$\partial_z \phi = \mathcal{V}\phi, \qquad \phi(z=0,\boldsymbol{x}) = \phi_0(\boldsymbol{x})$$

can be solved analytically because it is an ordinary differential equation:

$$\phi(z,\boldsymbol{x}) = e^{\int_0^z \mathcal{V}(z')dz'}\phi_0(\boldsymbol{x}) \quad = \quad \phi_0(\boldsymbol{x})\exp\Big(\frac{ik}{2}\int_0^z \mu(z',\boldsymbol{x})dz'\Big)$$

We cannot solve

$$\partial_z \phi = \mathcal{L}\phi + \mathcal{V}\phi, \qquad \phi(z=0,\boldsymbol{x}) = \phi_0(\boldsymbol{x})$$

analytically because $\mathcal{L}$ and $\mathcal{V}$ do not commute.

Numerical method for $z_n = nh$ and $\phi_n \simeq \phi(z_n, \boldsymbol{x})$:

$$\phi_{n+1} = e^{\mathcal{L}h} e^{\int_{z_n}^{z_{n+1}} \mathcal{V}(z)dz} \phi_n$$

In fact use the version:

$$\phi_{n+1} = e^{\mathcal{L}h/2} e^{\int_{z_n}^{z_{n+1}} \mathcal{V}(z)dz} e^{\mathcal{L}h/2} \phi_n$$

because

$$e^{(A+B)h} = e^{Ah} e^{Bh} + O(h^2)$$

while

$$e^{(A+B)h} = e^{Ah/2} e^{Bh} e^{Ah/2} + O(h^3)$$

By concatenating:

$$\phi_N = e^{\mathcal{L}h/2} e^{\int_{z_{N-1}}^{z_N} \mathcal{V}(z)dz} e^{\mathcal{L}h} e^{\int_{z_{N-2}}^{z_{N-1}} \mathcal{V}(z)dz} e^{\mathcal{L}h} \cdots e^{\mathcal{L}h} e^{\int_{z_0}^{z_1} \mathcal{V}(z)dz} e^{\mathcal{L}h/2} \phi_0$$